Multiscale/fractional step schemes for the numerical simulation of the rotating shallow water flows with complex periodic topography

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Abstract

In this paper, we study several multiscale/fractional step schemes for the numerical solution of the rotating shallow water equations with complex topography. We consider the case of periodic boundary conditions (\(f\)-plane model). Spatial discretization is obtained using a Fourier spectral Galerkin method. For the schemes presented in this paper we consider two approaches. The first approach (multiscale schemes) is based on topography scale separation and the numerical time integration is function of the scales. The second approach is based on a splitting of the operators, and the time integration method is function of the operator considered (fractional step schemes). The numerical results obtained are compared with the explicit reference scheme (Leap-Frog scheme). With these multiscale/fractional step schemes the objective is to propose new schemes giving numerical results similar to those obtained using only one uniform fine grid \(N \times N\) and a time step \(\Delta t\), but with a CPU time near the CPU time needed when using only one coarse grid \(N_1 \times N_1, N_1 < N\) and/or a time step \(\Delta t' > \Delta t\).

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1. Introduction: motivation of the problem

In full atmospheric models the influence of topography on temperature and precipitation is important. For example it is crucial to take correctly into account small scales of complex topography in order to obtain a good estimation of the total precipitation (see Smith [32], Smith and Barstad [33]) and the proper specification of temperature at the height of the topography is critical for the determination of snowpack and water resources. Primarily for these two reasons regional models are of use in climate simulations, so that the local impact of high resolution topographic forcing can be incorporated in simulations of seasonal climate and climate change predictions.

Despite this need, few studies have examined the efficacy of alternative representations of the dynamic effects of small scale topography on precipitation, the most notable exceptions being (Smith and Barstad [33], Leung and Ghan [34]). This is in spite of the fact that at least one component of the topographic forcing is known to high accuracy, the topography itself.

In recent years several new techniques have arisen within the applied mathematics community, that are able to utilize small scale information in a self-consistent fashion to approximate the effects of small scales on the temporal tendencies of...
the large scale. We report here on nascent attempts to apply these new methods to the problem of topographically forced flow and its dynamical and thermodynamical effects on the large scale. To test these methods we first use the simplest geophysical model that can adequately expose the dynamical effects of topography, a rotating shallow water system.

For the application of multiscale methods in meteorological modeling we can also refer to Vater et al. [37].

The shallow water equations with Coriolis forces are used in atmospheric modeling as a simplification of the primitive equations of atmospheric flow. In realistic applications there is variable bottom topography that adds a source term to the shallow water equations, similar to the Coriolis force. To take into account the small scales in the numerical computation of the solution of the shallow water problem with complex topography, it is necessary to retain sufficient modes (points) in the two spatial directions, thus increasing the computational time.

This article can be considered as a continuation of Dubois et al. [18] in which we proposed new numerical schemes to compute the rotating shallow water equations on a flat bottom. The topography $Z$ is introduced in the present study, which produces a perturbation term in the height equation (see the second equation (1.1)), with a separation of scales in the topography to distinguish between the large and small scales of the terrain (see (1.2) below). From the algorithmic point of view the reference scheme is the same, namely a spectral Galerkin method with an explicit time scheme. A more detailed comparison of the multilevel algorithms considered here and in Dubois et al. [18] appears below. Besides, as explained above (and below), we want here to produce decomposition methods based on the scales of the topography.

These new schemes are based on a scale separation (multiscale schemes) or on a splitting of the operators (fractional step schemes). The objective of this work is to propose new schemes giving numerical results close to those obtained using only one uniform fine grid $N \times N$ and a time step $\Delta t$, but with a CPU time comparable to the CPU time needed when using only one coarse grid $N_1 \times N_1$, $N_1 < N$ and/or a time step $\Delta t' > \Delta t$.

Let us consider the rotating shallow water equations with topography. The equations are written as follows:

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{Q} \times \mathbf{u}) + f \mathbf{u}^\perp + \nabla \left( gh + \frac{1}{2} |\mathbf{u}|^2 \right) &= 0, \\
\frac{\partial h}{\partial t} + \text{div}(h \mathbf{u}) &= \text{div}(Z_S \mathbf{u}),
\end{aligned}$$

(1.1)

where $\mathbf{u} = (u, v)^T$ is the velocity field, $\mathbf{u}^\perp = (-v, u)^T$ the orthogonal velocity field, $\mathbf{Q} = \nabla \times \mathbf{u}$ the vorticity vector, $h$ the height of the free surface, $Z_S$ is the topography, $f$ is the Coriolis force, $g$ the gravity and $|\cdot|$ the Euclidean norm.

With a multiscale scheme, we intend to use a scale separation based on the scales of the terrain, and then to adapt the equations as function of the scales.

Let us write the scale decomposition of the topography:

$$Z_S = \bar{Z}_S + Z'_S,$$

(1.2)

where $\bar{Z}_S$ (resp. $Z'_S$) corresponds to the large (resp. small) scales of the terrain. The time evolution equation of $h$ in (1.1) can be rewritten since it is linear in $Z_S$:

$$\frac{\partial h}{\partial t} + \text{div}(h \mathbf{u}) = \text{div}(\bar{Z}_S \mathbf{u}) + \text{div}(Z'_S \mathbf{u}),$$

(1.3)

Here we consider that $\bar{Z}_S$ is associated with a large scale size in space $\Delta \mathbf{x}$ and $Z'_S$ with a small scale size $\Delta \mathbf{x}'$, with $\eta \gg 1$, in order to take into account the small scales of the terrain. Moreover we retain the hypothesis that $Z'_S$ is small in comparison with $\bar{Z}_S$ (this is the case, for example, in the scale separation using Fourier expansion). Let us consider the following model for $Z'_S$:

$$Z'_S(\mathbf{x}) = \varepsilon Z_S(\mathbf{x}^\varepsilon),$$

(1.4)

where $\varepsilon \ll 1$, $\mathbf{x}$ is the spatial scale associated with $\Delta \mathbf{x}$ and $\mathbf{x}^\varepsilon = \eta \mathbf{x}$, $\varepsilon$ and $\eta$ independent.

From the scale separation $Z_S = \bar{Z}_S + Z'_S$ we can consider a decomposition on $h$:

$$h = \bar{h} + h',$$

(1.5)

where $\bar{h}$ is associated with $\bar{Z}_S$ and $h'$ is associated with $Z'_S$, with the following equations for $\bar{h}$ and $h'$, deduced from (1.3):

$$\begin{aligned}
\frac{\partial \bar{h}}{\partial t} + \bar{h} \text{div}(\mathbf{u}) + \mathbf{u} \cdot \nabla \bar{h} &= \bar{Z}_S \text{div}(\mathbf{u}) + \mathbf{u} \cdot \nabla Z_S, \\
\frac{\partial h'}{\partial t} + h' \text{div}(\mathbf{u}) + \mathbf{u} \cdot \nabla h' &= Z'_S \text{div}(\mathbf{u}) + \mathbf{u} \cdot \nabla Z'_S.
\end{aligned}$$

(1.6)

We have that $h = \bar{h} + h'$ is solution of (1.3). Using (1.4), we can rewrite (1.6) in the following manner:
\[
\begin{aligned}
\frac{\partial \tilde{h}}{\partial t} + \tilde{h} \text{div}(\mathbf{u}) + \mathbf{u} \cdot \nabla \tilde{h} &= \tilde{Z}_S \text{div}(\mathbf{u}) + \mathbf{u} \cdot \nabla \tilde{Z}_S, \\
\frac{\partial h'}{\partial t} + h' \text{div}(\mathbf{u}) + \mathbf{u} \cdot \nabla h' &= O(\varepsilon),
\end{aligned}
\] 

(1.7)

with \(O(\varepsilon) = \varepsilon \tilde{Z}_S \text{div}(\mathbf{u}) + \varepsilon \eta \mathbf{u} \cdot \nabla \tilde{Z}_S\). Comparing the equation for \(\tilde{h}\) and the equation for \(h'\) in (1.7), we surmise that \(h'\), \(\nabla h'\) and \(\frac{\partial h'}{\partial t}\) decrease in norm with \(\varepsilon\). So to take into account this different behavior of \(\tilde{h}\) and \(h'\) in space and time, we propose multiscale schemes to compute the influence of the large and small scales of the topography on the velocity field \(\mathbf{u}\) and on the height \(h\). Considering that \(h'\) is small in comparison with \(\tilde{h}\), we propose to compute \(h'\) less accurately than \(\tilde{h}\), without causing an error too large on \(h = \tilde{h} + h'\).

This article is organized as follows. In Section 2 we describe the problem that we study and the first topography considered for the numerical test case. In Section 3 we present the reference scheme (spectral Galerkin method in space and Leap-Frog scheme for time integration of all the scales). In Section 4 we present multiscale schemes (namely \(S_1\), \(S_2\) and \(S_3\) schemes) and in Section 5 a fractional step scheme (namely \(S_4\) schemes).

The scheme \((S_1)\) in this article is not present in Dubois et al. [18] and a theoretical study for this scheme is presented below (see Propositions 1, 2 and 3). The scheme \((S_1)\) proposed in Dubois et al. [18] is not considered here since it does not allow to choose a time step much larger than that of the explicit reference scheme. The scheme \((S_2)\) of the present article corresponds to the scheme \((S_2)\) presented in Dubois et al. [18] and it has been modified to account for the topography. The scheme \((S_3)\) of the present article is not present in Dubois et al. [18] and the scheme \((S_4)\) of the present article (fractional step scheme) corresponds to the scheme \((S_3)\) of Dubois et al. [18], modified for shallow water equations with complex topography.

In Sections 6 and 7 we consider two others complex topographies considered for the numerical test cases and we present the numerical results obtained with these topographies. Finally, Section 8 contains some concluding remarks and indications on future developments.

2. Description of the problem considered

We consider the two-dimensional nonlinear rotating shallow water problem with topography (1.1) and periodic boundary conditions (doubly periodic \(f\)-plane). This problem is considered as a global model for the simulation of atmospheric (or oceanic) flows using a planar geometry.

Here we have considered the formulation of the shallow water problem with the following scalar dependent variables instead of the velocity vector \(\mathbf{u}\): the vorticity \(\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\) and the plane divergence \(\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\). So, the problem considered is written as follows:

\[
\begin{aligned}
\frac{\partial \omega}{\partial t} + \frac{\partial (\omega u)}{\partial x} + \frac{\partial (\omega v)}{\partial y} + f\delta &= 0, \\
\frac{\partial \delta}{\partial t} + \frac{\partial (\omega u)}{\partial y} - \frac{\partial (\omega v)}{\partial x} - f\omega + \Delta \left(gh + \frac{1}{2}\mathbf{u}^2\right) &= 0, \\
\frac{\partial h}{\partial t} + \text{div}(h\mathbf{u}) &= 0
\end{aligned}
\]

(2.1)

The computational domain considered is \(\mathcal{M} = (0, L_x) \times (0, L_y)\), with \(L_x = L_y = 6.31 \times 10^6\) m (earth radius), the period in the \(x\) and \(y\) directions. In the following for \(h\) we have retained

\[
h = \langle h \rangle + h^* \tag{2.2}
\]

with \(\langle h \rangle = H = 10^4\) m (troposphere), \(\langle \cdot \rangle\) is the spatial average and \(h^*\) is the height of the free surface around \(H\) (\(h^*\) is zero mean value). As for \(f\) and \(g\) we have retained \(f = 10^{-4}\) s\(^{-1}\) (\(f\) is considered as constant corresponding to the \(f\)-plane approximation) and \(g = 9.81\) m/s\(^2\).

The problem (2.1) induces substantial numerical difficulties if we want to compute directly the numerical approximation of (2.1). Indeed, most atmospheric flows are turbulent flows, i.e., they contain a wide range of scales with very different spatial size and characteristic times. To overcome these numerical difficulties, Large Eddy Simulation (LES) models for turbulence modeling are usually proposed, in order to compute only the large scales of the flow (which contain most of the kinetic energy and the enstrophy in two-dimensional turbulent flows), and modeling the dissipative action of the small scales (which are not computed) on the large ones (see, for example, Sagaut [31] and Piomelli [29] for more details). In meteorology, a model often used consists in adding a hyperdissipative operator in Eqs. (2.1) (see Basdevant et al. [3], Browning et al. [7], Browning and Kreiss [8], Pascal and Basdevant [27], Tadmor [36] and Gelb and Gleeson [22]). Such an operator is of the form:

\[
\nu_{T} \Delta^{2p}, \tag{2.3}
\]
with \( p \) an integer parameter and \( \nu_T \) the turbulent viscosity (or eddy viscosity):
\[
\nu_T = \frac{\zeta}{k_{\text{max}}^4 \Delta t};
\]  
(2.4)
here \( \Delta t \) is the time step retained for the numerical computation, \( k_{\text{max}} \) is the modulus of the highest wavenumber associated with the smallest computed scales: \( k_{\text{max}} = \sqrt{2 \pi} (\frac{\pi}{L}) = \sqrt{2 \pi} (\frac{\pi}{L_x}) \) where \( N \) is the total number of modes retained in each spatial direction (see Section 3), \( \zeta \) is a nondimensional positive constant and \( p \) is an integer. In practice, for the numerical simulations described in this article, we have chosen \( p = 2 \) and \( \zeta = 10^4 \).

As we will see in the next section, the role of the additional term (2.3) in the equations (2.1) is to prevent spectral reflections in the high wavenumbers of the spectra, spectral reflection due to the spatial discretization (truncated spectrum), in order to obtain an energy spectrum (velocity spectrum) with a slope of \( k^{-5} \) in the inertial range in agreement with the two-dimensional homogeneous turbulence theory (see Kraichnan [24]). The slope of the spectrum associated with the height is \( k^{-5} \), since \( h \) appears through a gradient in the velocity equation (1.1) (see Cushman-Roisin [16]). Finally, the problem considered here can be written as follows:
\[
\begin{align*}
\frac{\partial \varrho}{\partial t} + \nu_T \Delta^3 \varrho + \frac{\partial (\varrho u)}{\partial x} + \frac{\partial (\varrho v)}{\partial y} + f \varrho &= 0, \\
\frac{\partial \varrho}{\partial t} + \nu_T \Delta^3 \varrho + \frac{\partial (\varrho u)}{\partial y} - \frac{\partial (\varrho v)}{\partial x} - f \varrho + \Delta \left( g h + \frac{1}{2} | \mathbf{u} |^2 \right) &= 0, \\
\frac{\partial h}{\partial t} + \nu_T \Delta^3 h + \text{div}(h \mathbf{u}) &= \text{div}(Z_S \mathbf{u}).
\end{align*}
\]  
(2.5)

For the numerical simulations we consider the case of a flow past an idealized topography. We have considered a periodic topography in the two spatial directions \( x \) and \( y \). So we can replace the topography with its Fourier series, under some regularity hypothesis. And we approximate the topography considering a truncation of the Fourier series associated with the topography, since the Fourier coefficients decrease when the wavenumber \( k \) increases, and the decreasing is function of the regularity of the topography (more the topography is regular, more the decrease is fast). If we consider a small decrease of the Fourier coefficients (for example \( O(1/k) \)) when \( k \) increases and if we retain a number of Fourier coefficients sufficiently large, we can obtain a complex topography with different spatial scales.

Here we have considered the following complex topography periodic in space:
\[
Z_{S, N}(\mathbf{x}) = \mathcal{H} + \sum_{\mathbf{k} \in \mathbb{Z}_N^2} \frac{C}{k} \cos(\mathbf{k} \cdot \mathbf{x})
\]  
(2.6)
where \( \mathbf{x} \in \mathcal{M}, \mathbf{k} = (k_1, k_2), \mathbf{k}' = (k_1', k_2') = (\frac{2\pi}{L_x} k_1, \frac{2\pi}{L_y} k_2), \mathbf{x} = (x, y), \mathbf{k} \cdot \mathbf{x} = k_1' x + k_2' y \) is the Euclidean scalar product, \( k = |\mathbf{k}|, \mathcal{H} = 10^3 \) m, \( C = 100, \mathbb{J}_N = [0, N]^2 \subseteq \mathbb{Z}^2, \mathbb{I}_N = [0, N] - [0] \) and \( N = 64 \). In Fig. 1 we have represented the spectrum of \( Z_{S, N} \) (Fig. 1(a)) and the behavior of \( Z_{S, N} \) in the physical space (Figs. 1(b) to 1(d)). In the next sections (Sections 3 to 5), comparisons of the different schemes proposed are done using the topography (2.6).

3. The reference scheme

For the spatial discretization, we have used a spectral Galerkin method (see Gottlieb and Orszag [23], Canuto et al. [9]), with the trigonometric polynomials as Galerkin basis since the boundary conditions are periodic (Fourier spectral Galerkin method). Since \( \varrho, \delta \) and \( h \) are periodic in space, we can consider the infinite Fourier expansion for these dependent variables. If the dependent variables are regular, the Fourier coefficients \( \hat{\omega}_k, \hat{\delta}_k \) and \( \hat{h}_k \) decrease rapidly when \( |k| \) increases (see Gottlieb and Orszag [23] and Canuto et al. [9] for example), possibly exponentially fast (Gevrey regularity, see Foias and Temam [21]). So, we can look for an approximation of \( \varrho, \delta \) and \( h \) of the following form (truncated Fourier series expansions):
\[
\begin{align*}
\omega_N(\mathbf{x}, t) &= P_N \omega(\mathbf{x}, t) = \sum_{\mathbf{k} \in \mathbb{J}_N} \hat{\omega}_k(t) \exp(ik' \cdot \mathbf{x}), \\
\delta_N(\mathbf{x}, t) &= P_N \delta(\mathbf{x}, t) = \sum_{\mathbf{k} \in \mathbb{J}_N} \hat{\delta}_k(t) \exp(ik' \cdot \mathbf{x}), \\
h_N(\mathbf{x}, t) &= P_N h(\mathbf{x}, t) = \sum_{\mathbf{k} \in \mathbb{J}_N} \hat{h}_k(t) \exp(ik' \cdot \mathbf{x}),
\end{align*}
\]  
(3.1)
with \( P_N \) the orthogonal projection on the space \( V_N = \text{Span}(\exp(ik' \cdot \mathbf{x}), \mathbf{k} \in \mathbb{J}_N) \) and \( \mathbb{J}_N = [1 - N/2, N/2]^2 \subseteq \mathbb{Z}^2 \). The total number of modes retained is \( N^2 \). We have retained the same number of modes \( N \) in the two spatial directions \( x \) and \( y \), since the computational domain \( \mathcal{M} \) is quadratic \( (L_x = L_y) \), and the turbulence is homogeneous and isotropic (see Batchelor [4]).
Moreover, we note that for $\omega_N$, $\delta_N$ and $h_N$ defined in (3.1), the periodic boundary conditions are automatically satisfied (Galerkin approximation).

The Fourier coefficients are computed using a Method of Weighted Residuals (MWR, see for example Gottlieb and Orszag [23], Canuto et al. [9]). We impose that the residuals, obtained by substituting $\omega$, $\delta$, $h$ and $u$ with $\omega_N$, $\delta_N$, $h_N$ and $u_N$, in (2.5), have an orthogonal projection $P_N$ on the space $V_N$ equal to zero, for the scalar product $(\cdot, \cdot)_{L^2(M)}$ defined in $L^2(M)$. This is equivalent to minimizing the residuals in the energy norm (least squares method), i.e. for the norm $|\cdot|_{L^2(M)}$ associated with the previous scalar product. So, using the orthogonality properties of the Fourier polynomials for the $L^2(M)$ scalar product, we obtain the following system of Ordinary Differential Equations (ODEs), for the Fourier coefficients $\hat{\omega}_k$, $\hat{\delta}_k$ and $\hat{h}_k$, $k \in \mathbb{N}$:
\[
\begin{align*}
\frac{d}{dt} \hat{\omega}_k + v_T |k'|^4 \hat{\omega}_k + \hat{T}_{\omega,N}(k) &= \hat{\omega}_{\omega, N}(k), \\
\frac{d}{dt} \hat{\delta}_k + v_T |k'|^4 \hat{\delta}_k + \hat{T}_{\delta,N}(k) &= \hat{\delta}_{\delta, N}(k), \\
\frac{d}{dt} \hat{h}_k + v_T |k'|^4 \hat{h}_k + \hat{T}_{h,N}(k) &= \hat{h}_{h, N}(k),
\end{align*}
\] (3.2)

We have denoted by \( T_{\omega,N}, T_{\delta,N} \) and \( T_{h,N} \) the approximations of the nonlinear convective terms:

\[
\begin{align*}
T_{\omega,N} &= \frac{\partial}{\partial x} (\omega_N u_N) + \frac{\partial}{\partial y} (\omega_N v_N), \\
T_{\delta,N} &= \frac{\partial}{\partial y} (\omega_N u_N) - \frac{\partial}{\partial x} (\omega_N v_N) + \frac{1}{2} \Delta (|u_N|^2), \\
T_{h,N} &= \text{div}(h_N u_N),
\end{align*}
\] (3.3)

and by \( F_{\omega,N}, F_{\delta,N} \) and \( F_{h,N} \) the approximations of the (source) terms:

\[
\begin{align*}
F_{\omega,N} &= -f \hat{\delta}_N, \\
F_{\delta,N} &= f \omega_N - g \Delta h_N, \\
F_{h,N} &= \text{div}(Z_S u_N).
\end{align*}
\] (3.4)

We can obtain for \( F_{h,N} \) a good approximation of \( F_h = \text{div}(Z_S u) \). Indeed, the Fourier coefficients \( \hat{Z}_{S,N}(k) \) of the first topography (2.6) decrease as \( O(1/k) \), so \( \nabla \hat{Z}_{S,N}(k) \) decreases as \( O(1) \) when \( k \) increases. However, in the source terms (3.4) the topography appears in the term \( \text{div}(Z_{S,N} u_N) \). If we assume a turbulent two dimensional flow, we have a decrease of the energy spectrum associated with \( u_N \) as \( k^{-3} \) in the inertial range, and faster in the dissipative range. So the Fourier coefficients associated with the source term \( \text{div}(Z_{S,N} u_N) \) decrease when \( k \) increases and when we use a spectral approximation in space we have convergence when \( N \) increases.

With spectral Galerkin methods, one of the difficulties is the computation of the nonlinear terms. For the computation of the Fourier coefficients \( T_{\omega,N}(k), T_{\delta,N}(k) \) and \( T_{h,N}(k) \) of the nonlinear terms, we use a pseudospectral method [see Gottlieb and Orszag [23], Canuto et al. [9]]. In this way, the total number of operations required to compute the nonlinear terms is \( O(N^2 \log_2(N)) \) operations. The evaluation of the Fourier coefficients of the three nonlinear terms (3.3) of the shallow water problem (2.5), requires 9 FFTs (direct and inverse), at each time step. To eliminate the aliasing error we use the 3/2 rule (see Canuto et al. [9]).

For the time integration of the previous system of ODEs (3.2), we shall consider an explicit scheme. The trigonometric polynomials being eigenfunctions of the hyperdissipative operator (2.3) used for the turbulence modeling, the matrix obtained is diagonal. So an exact time integration of the dissipative terms can be obtained, with no restrictive stability constraint (see Canuto et al. [9], Dubois and Jauberteau [17]).

Classically in meteorology, the Leap-Frog scheme, which is a second order explicit scheme, is used for the time integration of the rotation, convective and gravity terms (see Durran [19], Williamson and Laprise [39] for example). In the present case, we obtain, \( \forall k \in \mathbb{N} \):

\[
\begin{align*}
\hat{\omega}^{n+1}_k &= \exp(-2v_T \Delta t |k'|^4p) \hat{\omega}^n_k - 2\Delta t \exp(-v_T \Delta t |k'|^4p) (\hat{T}_{\omega,N}(k) - \hat{T}_{\omega,N}(k)), \\
\hat{\delta}^{n+1}_k &= \exp(-2v_T \Delta t |k'|^4p) \hat{\delta}^n_k - 2\Delta t \exp(-v_T \Delta t |k'|^4p) (\hat{T}_{\delta,N}(k) - \hat{T}_{\delta,N}(k)), \\
\hat{h}^{n+1}_k &= \exp(-2v_T \Delta t |k'|^4p) \hat{h}^n_k - 2\Delta t \exp(-v_T \Delta t |k'|^4p) (\hat{T}_{h,N}(k) - \hat{T}_{h,N}(k)),
\end{align*}
\] (3.5)

where \( \Delta t \) is the time step retained for the time integration, \( \hat{\omega}^{n+1}_k, \hat{\delta}^{n+1}_k \) and \( \hat{h}^{n+1}_k \) are approximations of \( \hat{\omega}_k(t_{n+1}), \hat{\delta}_k(t_{n+1}) \) and \( \hat{h}_k(t_{n+1}) \) respectively. We can see in (3.5) the effect of the additional term (2.3): it is the multiplication of the unknowns with \( \exp(-2v_T \Delta t |k'|^4p) \in [0,1] \) and of the nonlinear terms and source terms with \( \exp(-v_T \Delta t |k'|^4p) \in [0,1] \). These exponential multiplicative terms decrease when \( |k'| \) increases, and the decrease is amplified with the choice of the parameter \( p \) in (2.3). So, when \( |k'| \) increases, the Fourier coefficients \( \hat{\omega}^{n+1}_k, \hat{\delta}^{n+1}_k \) and \( \hat{h}^{n+1}_k \) of \( \hat{\omega}_N^{n+1}, \hat{\delta}_N^{n+1} \) and \( \hat{h}_N^{n+1} \) are damped more and more when \( |k'| \) increases, avoiding spectral accumulation and reflection of the energy on the high wavenumbers, due to the nonlinear terms.

Using the classical von Neumann stability analysis, we obtain the stability constraint for this explicit scheme (see Roache [30], Williamson and Laprise [39]):

\[
|U \pm \sqrt{g(H - \langle Z_S \rangle) + f^2 \frac{1}{k'^2}}| \Delta t k' \leq 1,
\] (3.6)

where \( k' = |k'| \), \( U \) is a constant advecting velocity (linearized problem) and \( H - \langle Z_S \rangle = (h - Z_S) \) is the spatial average value of \( h - Z_S \).
Hereafter this spectral Galerkin method, with Leap-Frog scheme to compute all the scales, will be considered as reference scheme \((S_{\text{Ref}}^{N,\Delta t})\). We have retained \(N = 64\) and \(\Delta t = 10\) s in order to obtain numerical stability and convergence. The mesh size is \(\Delta x = L_x/N = L_y/N\). The numerical validation of the new multiscale/fractional step schemes proposed will be done comparing with this reference simulation.

The choice of the initial condition is obtained firstly by imposing an energy spectrum decreasing as \(k^{-3}\) (two dimensional turbulent flow), and with a plane divergence \(\delta\) equal to zero (barotropic flow). The phases of the Fourier coefficients of the initial condition are random numbers. We run a large number of time iterations until reaching a statistically steady state, i.e. the averages of global quantities in space are approximately time independent. Then, we use this velocity field obtained to start the comparison between the new proposed schemes and the reference explicit scheme. For more details, see Dubois and Jauberteau [17]. Since the explicit Leap-Frog scheme required two initial time steps, the first time step is obtained using the enstrophy \(\|\omega\|_{L^2(\mathcal{M})}\) (or kinetic energy) and of the height \(\|h\|_{L^2(\mathcal{M})}\):

\[
\|\mathbf{u}\|_{L^2(\mathcal{M})}^2 = \int_{\mathcal{M}} |\mathbf{u}(x)|^2 \, dx, \quad \|h\|_{L^2(\mathcal{M})}^2 = \int_{\mathcal{M}} |h(x)|^2 \, dx,
\]

(3.7)

the enstrophy \(\|\mathbf{u}\|_{H^1(\mathcal{M})}\) (i.e. the energy norm of the gradient of the velocity field):

\[
\|\mathbf{u}\|_{H^1(\mathcal{M})}^2 = \int_{\mathcal{M}} |\nabla \mathbf{u}(x)|^2 \, dx,
\]

(3.8)

the maximum value of the velocity field \(\|\mathbf{u}\|_{L^\infty(\mathcal{M})}\) and of the height \(\|h\|_{L^\infty(\mathcal{M})}\):

\[
\|\mathbf{u}\|_{L^\infty(\mathcal{M})} = \sup_{x \in \mathcal{M}} |\mathbf{u}(x)|, \quad \|h\|_{L^\infty(\mathcal{M})} = \sup_{x \in \mathcal{M}} |h(x)|.
\]

(3.9)

Firstly we compare \((S_{\text{Ref}}^{N,\Delta t})\) and \((S_{\text{Ref}}^{N_1,\Delta t})\) for \(N = 64\) and \(N_1 = 32\). In each case we have retained the same time step \(\Delta t = 10\) s. In Fig. 2 we have represented the velocity \(\mathbf{u} = (u, v)\) and the height \(h\) computed with \((S_{\text{Ref}}^{N,\Delta t})\) and \((S_{\text{Ref}}^{N_1,\Delta t})\). This comparison shows the effect of the high modes \(k\), for \(32 < k \leq 64\), on the computation of the velocity field and of the height. We can see that the curves for the numerical results obtained with \((S_{\text{Ref}}^{N,\Delta t})\) have less oscillations than the ones associated with \((S_{\text{Ref}}^{N_1,\Delta t})\) (regularization).

Comparing, on the global quantities (3.7)–(3.9), \((S_{\text{Ref}}^{N,\Delta t})\) and \((S_{\text{Ref}}^{N_1,\Delta t})\), the time average of the relative error is equal to 0.43 on the kinetic energy, 0.51 on the enstrophy and 0.33 on the maximum value of the velocity. As for the height, the relative error in energy norm is equal to 0.46 and 0.34 on the maximum value. Time averages are computed over all the time interval.

4. New schemes based on a splitting of the topography scales (multiscale schemes)

The system of equations (3.2) can be rewritten:

\[
\begin{align*}
\frac{\partial \omega_N}{\partial t} + v_T \Delta^2 \omega_N + P_N T_{\omega, N} = P_N F_{\omega, N}, \\
\frac{\partial \delta_N}{\partial t} + v_T \Delta^2 \delta_N + P_N T_{\delta, N} = P_N F_{\delta, N}, \\
\frac{\partial h_N}{\partial t} + \Delta^2 \frac{1}{\Delta t} h_N + P_N T_{h, N} = P_N F_{h, N}.
\end{align*}
\]

(4.1)

Following (1.2), we consider the scale decomposition of the topography \(Z_{S,N}\):

\[
Z_{S,N} = Z_{S,N_1} + Z_{S,N_1}^N,
\]

(4.2)

with \(N\) associated with the fine grid \(N \times N\), and \(N_1 < N\) associated with the coarse grid \(N_1 \times N_1\), \(Z_{S,N_1} = P_{N_1} Z_{S,N}\) and \(Z_{S,N_1}^N = Q_{N_1}^N Z_{S,N}\), where \(Q_{N_1}^N = I_N - P_{N_1}\), with \(I_N\) the identity operator (identity square matrix of order \(N\)). The scale decomposition (4.2) on \(Z_{S,N}\) induces a scale decomposition on \(U_N = (\omega_N, \delta_N, h_N)\). Indeed using the projection operators
Fig. 2. Numerical comparison of \((S_{\text{Ref}}^{N,\Delta t})\) and \((S_{\text{Ref}}^{N_1,\Delta t})\) for \(N = 64, N_1 = 32\) and \(\Delta t = 10\) s. The comparison is done on the computation of the velocity field \(u = (u, v)\) and the height \(h^*\) with \(h = H + h^*\).

\(P_{N_1}\) and \(Q_{N_1}\), we can write \(U_N = U_{N_1} + U_{N_1}^N\), where \(U_{N_1}\) satisfies the following equations:

\[
\begin{align*}
\frac{\partial \omega_{N_1}}{\partial t} + \nu T \Delta^2 p \omega_{N_1} + P_{N_1} T_{\omega,N_1} + P_{N_1} T_{\omega,N_1}^N &= P_{N_1} F_{\omega,N_1} (= F_{\omega,N_1}), \\
\frac{\partial \delta_{N_1}}{\partial t} + \nu T \Delta^2 p \delta_{N_1} + P_{N_1} T_{\delta,N_1} + P_{N_1} T_{\delta,N_1}^N &= P_{N_1} F_{\delta,N_1} (= F_{\delta,N_1}), \\
\frac{\partial h_{N_1}}{\partial t} + \nu T \Delta^2 p h_{N_1} + P_{N_1} T_{h,N_1} + P_{N_1} T_{h,N_1}^N &= P_{N_1} F_{h,N_1} (= F_{h,N_1}).
\end{align*}
\]

(4.3)

and \(U_{N_1}^N\) satisfies:

\[
\begin{align*}
\frac{\partial \omega_{N_1}^N}{\partial t} + \nu T \Delta^2 p \omega_{N_1}^N + Q_{N_1}^N T_{\omega,N} &= Q_{N_1}^N F_{\omega,N} (= F_{\omega,N_1}), \\
\frac{\partial \delta_{N_1}^N}{\partial t} + \nu T \Delta^2 p \delta_{N_1}^N + Q_{N_1}^N T_{\delta,N} &= Q_{N_1}^N F_{\delta,N} (= F_{\delta,N_1}), \\
\frac{\partial h_{N_1}^N}{\partial t} + \nu T \Delta^2 p h_{N_1}^N + Q_{N_1}^N T_{h,N} &= Q_{N_1}^N F_{h,N} (= F_{h,N_1}).
\end{align*}
\]

(4.4)

In (4.3) the nonlinear coupling terms between the large and small scales appear as:

\[
\begin{align*}
T_{\omega,N_1}^N &= T_{\omega,N} - T_{\omega,N_1}, \\
T_{\delta,N_1}^N &= T_{\delta,N} - T_{\delta,N_1}, \\
T_{h,N_1}^N &= T_{h,N} - T_{h,N_1}.
\end{align*}
\]

(4.5)

The nonlinear coupling terms (4.5) reflect the action of the small scales on the large ones, and in particular the action of the small scales contained in the topography on the large scales of the flow through the nonlinear coupling term \(T_{h,N_1}^N\). In Bresch et al. [6], the authors performed multiple scale analyses of the shallow water equations, showing an influence of the
small scale topography on the large scale flow. So these nonlinear terms should be taken into account in the multiscale schemes proposed in this section. This will be the case as will be seen below (closure problem).

In the spectral case we have the following correspondence for $\tilde{h}$ and $h'$ (see (1.5)) : $\tilde{h}$ corresponds to $P_N h$ which is present in $F_{kN}$ and so in the equation of $\delta_{N1}$, i.e. in the equation associated with $U_{kN}$; $h'$ corresponds to $Q_{kN} N h$ which is present in $I_{kN}$ (see the second equation in (4.4)) and so in the equation of $\delta_{N1}$: $h'$ is not directly in the equation associated with $U_{kN}$, but not directly in the equation of $U_{N1}$. Since, in norm, $h_{N1}$ (resp. $U_{N1}$) is small in comparison with $h_{N1}$ (resp. $U_{N1}$) the influence of $h_{N1}$ in $U_{N1}$ is limited.

The multiscale schemes described in this paragraph consist in computing differently the large scales (4.3) and the small scales (4.4) to take into account the different dynamics of the large and small scales. They can be compared with multirate time integration schemes (Constantinescu and Sandu [15]). However, here the adaptation is done in the spectral space, not in the physical space. We present hereafter the numerical study of these different multiscale schemes.

4.1. Multiscale scheme ($S_1$): quasistatic approximation

We consider the high modes. Since, in norm, the time derivative of the high modes (associated with the small scales) is smaller than the time derivative of the low modes (associated with the large scales) we intend to update less often the coefficients associated with the high modes $k \in \mathbb{Z} \setminus \mathcal{N}_l$, with $N = 64$ and $N_1 = 32$ (small scales) and the nonlinear interactions between the high and low modes differently than we do for the coefficients of the low modes. For the scheme ($S_1$) at each time step $\Delta t$, where $\Delta t$ is the same time step as for the reference scheme, we evaluate $U_{N1} = P_{N1} U$ (associated with the large scales). As for the small scales $U_{N1} = U_{N} - U_{N1}$, since $\frac{\partial U_{N1}}{\partial t}$ is small in norm, we use a quasistatic approximation to evaluate $U_{N1}$ i.e. we consider that $U_{N1}$ is constant in time over $l$ time steps, $l > 1$. So at time $t_n$ we compute $U_{N1}$, i.e. $U_{N1}$ and $U_{N1}$ on the fine level with (4.1). Then, over $l$ time steps, we compute only $U_{N1}$ on the coarse level with (4.3) and we keep $U_{N1}$ constant (quasistatic approximation) like for the nonlinear interaction terms $P_{N1} T_{N1}^N$, $P_{N1} T_{N1}^N$ and $P_{N1} T_{N1}^{N \cdot h}$, (closure problem) present in the equation of $U_{N1}$ (see (4.3) and (4.5)). Then, at time $(n + l) \Delta t$ we compute $U_{N1}$ over the fine level with (4.1), i.e. $U_{N1}$ and $U_{N1}$ and we update the nonlinear interaction terms.

So we obtain at each time step $\Delta t$ a prediction of the velocity $u = P_N (u)$ and height $h = P_N (h)$ on the fine grid computing $u_{N1} = P_{N1} u$ and $h_{N1} = P_{N1} h$ on the coarse grid. The correction is obtained updating $Q_{N1} N u$ and $Q_{N1} N h$, less frequently (on every $(l + 1)$ time steps i.e. $(l + 1) \Delta t$, $l > 0$) on the fine grid $N \times N$. As for the nonlinear coupling terms between the large and small scales appearing in the equations on the coarse grid the closure is obtained keeping them constant during $l$ time steps. We obtain the quasistatic scheme ($S_{11}^{N_1}$) and we can write:

$$\mathbf{U}_{N}^{n+l} = \mathbf{U}_{N1}^{n+l} + \mathbf{U}_{N1}^{n,ml},$$

for $k \in [1, l] \in \mathbb{Z}$. Following (1.5), (4.6) can be considered as a scale decomposition of the unknown $\mathbf{U} = \tilde{\mathbf{U}} + \mathbf{U}'$, where $\tilde{\mathbf{U}}$ is a prediction value ($\mathbf{U}_{N1}^{n+l}$), updated at each time step and $\mathbf{U}'$ is a correction value ($\mathbf{U}_{N1}^{n,ml}$), updated every $(l + 1)$ time step.

Now we look for an estimation of the error due to the quasistatic approximation. We consider the following equation:

$$\frac{\partial U_{N1}^{n}}{\partial t} = Q_{N1} N \mathbf{L}(U_{N1}, U_{N1}),$$

where $\mathbf{L}(\cdot, \cdot)$ is a nonlinear term which depends on the variables $U_{N1}$ and $U_{N1}$. We suppose that the nonlinear term is bilinear i.e. linear with respect to $U_{N1}$ and with respect to $U_{N1}$, as it is the case for the nonlinear terms (3.3).

We want to compare, over one iteration, the small scales $\mathbf{U}_{N1}^{n+1}$ computed with (S1) or with (S11) to the small scales computed with the reference scheme ($S_{Ref}^{N_1}$). For this we suppose that at the previous times $(n - l) \Delta t$, $l \geq 0$, $\mathbf{U}_{N1}^{n-l}$ is exact, i.e. that $\mathbf{U}_{N1}^{n-l} = U_{N}(n-l)$. We consider the numerical approximation of $U_{N1}^{n}$ using an explicit scheme (Euler scheme for simplicity) with a time step $\Delta t$. We obtain:

$$\tilde{\mathbf{U}}_{N1}^{n+1} = \tilde{\mathbf{U}}_{N1}^{n} + \Delta t Q_{N1} N \mathbf{L}(\tilde{\mathbf{U}}_{N1}, \tilde{\mathbf{U}}_{N1}).$$

(4.8)

Now, if we consider a quasistatic approximation over $l$ time steps for the coefficients associated with the high modes (small scales), we assume that $\tilde{\mathbf{U}}_{N1}^{n} \approx \bar{\mathbf{U}}_{N1}^{n-l}$. So (4.8) can be written:

$$\bar{\mathbf{U}}_{N1}^{n+1} = \bar{\mathbf{U}}_{N1}^{n-l} + \Delta t Q_{N1} N \mathbf{L}(\bar{\mathbf{U}}_{N1}, \bar{\mathbf{U}}_{N1}).$$

(4.9)

We can rewrite (4.9) as follows: there exists $\xi \in (t_{n-l}, t_n)$ such that

$$\bar{\mathbf{U}}_{N1}^{n+1} = \bar{\mathbf{U}}_{N1}^{n-l} - l \Delta t \frac{\partial U_{N1}^{n}}{\partial t}(\xi) + \Delta t Q_{N1} N (\mathbf{L}(\bar{\mathbf{U}}_{N1}, \bar{\mathbf{U}}_{N1}) - \mathbf{L}(\bar{\mathbf{U}}_{N1}, \mathbf{U}_{N1})) + \mathbf{L}(\mathbf{U}_{N1}, \mathbf{U}_{N1}).$$

(4.10)
So we obtain:
\[
\tilde{U}_{N_1}^{n+1} = U_{N_1}^n + \Delta t Q_{N_1}^N (U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) - \Delta t \frac{\partial U_{N_1}^n}{\partial t}(\xi) + \Delta t Q_{N_1}^N \left( \text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) - \text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) \right).
\]
(4.11)
Using (4.8) we deduce:
\[
\tilde{U}_{N_1}^{n+1} = U_{N_1}^n + \Delta t \frac{\partial U_{N_1}^n}{\partial t}(\xi) + \Delta t Q_{N_1}^N \left( \text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) - \text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) \right).
\]
(4.12)
In the difference of the nonlinear terms in (4.12), only the nonlinear terms depending on the small scales remain. We make the assumption that \(\text{NL}(U_{N_1}, \tilde{U}_{N_1})\) is a K-lipschitz function with respect to its second variable. This will be the case if, in the nonlinear terms, we neglect the second order nonlinear terms, i.e. the terms which depend only on the small scales. We obtain the following majoration:
\[
|\tilde{U}_{N_1}^{n+1} - \tilde{U}_{N_1}^{n+1}|_{L^2(\mathcal{M})} \leq \Delta t \left\| \frac{\partial U_{N_1}^n}{\partial t} \right\|_{L^\infty([0, T]; L^2(\mathcal{M}))} \quad \text{where} \quad K = K(U_{N_1}^n). \quad \text{(4.13)}
\]
So we deduce from (4.17) and (4.18) the following error estimate:

**Proposition 1.**
\[
|\tilde{U}_{N_1}^{n+1} - \tilde{U}_{N_1}^{n+1}|_{L^2(\mathcal{M})} \leq \Delta t (1 + K \Delta t) \left\| \frac{\partial U_{N_1}^n}{\partial t} \right\|_{L^\infty([0, T]; L^2(\mathcal{M}))}. \quad \text{(4.14)}
\]
So, the error due to the quasistatic approximation will be small if the time derivative of the coefficients associated with the high modes are small (this is consistent with the hypothesis done in (1.7)). The error estimate will be around \(O(\Delta t)\) if \(\left\| \frac{\partial U_{N_1}^n}{\partial t} \right\|_{L^\infty([0, T]; L^2(\mathcal{M}))} \approx 1\).

Now we look for an estimation of the error using the explicit scheme (4.8) for the small scales, with a time step \((l+1)\Delta t\) instead of \(\Delta t\). We have:
\[
\tilde{U}_{N_1}^{n+1} = U_{N_1}^n + (l+1) \Delta t Q_{N_1}^N (U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) \quad \text{(4.15)}
\]
The difference between (4.15) and (4.8) gives:
\[
\tilde{U}_{N_1}^{n+1} - \tilde{U}_{N_1}^{n+1} = U_{N_1}^n - \tilde{U}_{N_1}^{n, n} + (l+1) \Delta t Q_{N_1}^N \left( \text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) - \Delta t Q_{N_1}^N \right).
\]
So there exists \(\xi \in (t_{n-1}, t_n)\) such that
\[
\tilde{U}_{N_1}^{n+1} - \tilde{U}_{N_1}^{n+1} = -\Delta t \frac{\partial U_{N_1}^n}{\partial t}(\xi) + (l+1) \Delta t Q_{N_1}^N \left( \text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) - \Delta t Q_{N_1}^N \right).
\]
As previously, in the nonlinear terms we neglect the second order nonlinear terms. So we can write:
\[
\tilde{U}_{N_1}^{n+1} - \tilde{U}_{N_1}^{n+1} = -\Delta t \frac{\partial U_{N_1}^n}{\partial t}(\xi) + (l+1) \Delta t Q_{N_1}^N \left( \text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n}) - \Delta t Q_{N_1}^N \right).
\]
(4.16)
As for Proposition 1, using the fact that \(\text{NL}(U_{N_1}^n, \tilde{U}_{N_1}^{n, n})\) is a K-lipschitz function with respect to its second variable, since we neglect the second order nonlinear terms, we obtain:
\[
|\tilde{U}_{N_1}^{n+1} - \tilde{U}_{N_1}^{n+1}|_{L^2(\mathcal{M})} \leq \Delta t \left\| \frac{\partial U_{N_1}^n}{\partial t} \right\|_{L^\infty([0, T]; L^2(\mathcal{M}))} \quad \text{where} \quad K = K(U_{N_1}^n) \quad \text{(4.17)}
\]
However:
\[
|\tilde{U}_{N_1}^{n+1} - (l+1) \tilde{U}_{N_1}^{n+1}|_{L^2(\mathcal{M})} \leq \Delta t \left\| \frac{\partial U_{N_1}^n}{\partial t} \right\|_{L^\infty([0, T]; L^2(\mathcal{M}))}
\]
As (4.17) and (4.18) the following error estimate:

**Proposition 2.**
\[
|\tilde{U}_{N_1}^{n+1} - U_{N_1}^n|_{L^2(\mathcal{M})} \leq \Delta t (1 + K \Delta t) \left\| \frac{\partial U_{N_1}^n}{\partial t} \right\|_{L^\infty([0, T]; L^2(\mathcal{M}))} + K \Delta t |\tilde{U}_{N_1}^{n, n}|_{L^2(\mathcal{M})}. \quad \text{(4.19)}
\]
Remark 1. We can see that the error estimate (4.14) (quasistatic approximation over $l$ time steps $\Delta t$) is better than the error estimate (4.19) (explicit scheme with a time step $(l + 1)\Delta t$). This will be numerically illustrated hereafter.

In Fig. 3 we have compared the velocity field $u = (u, v)$ and the height $h$ computed with $(S_{\text{Ref}}^{N,\Delta t})$, $(S_{\text{Ref}}^{N,\Delta t'})$ and $(S_{1,1,1}^{N,\Delta t})$ for $N = 64$, $N_1 = 32$, $\Delta t = 10$ s and $\Delta t' = \Delta t$. For the value of the parameter $l$ of $(S_{1,1,1}^{N,\Delta t})$ (see (4.6)) we have retained $l = 5$. Comparing with $(S_{\text{Ref}}^{N,\Delta t})$, we can see in Fig. 2 and Fig. 3 a significant improvement of the accuracy on the computation of $u$ and $h$ with $(S_{1,1,1}^{N,\Delta t})$. The CPU time consumed with $(S_{1,1,1}^{N,\Delta t})$ is intermediary between the CPU time consuming for $(S_{\text{Ref}}^{N,\Delta t})$ and the one consuming for $(S_{1,1,1}^{N,\Delta t})$.

Note that $(S_{1,1,1}^{N,\Delta t})$ and $(S_{\text{Ref}}^{N,\Delta t'})$, for $\Delta t' = \Delta t$, are similar in CPU time. Indeed most of the CPU time is consumed in the FFTs computation on the fine level $N$ and the number of FFTs computed on the fine level is the same for the two considered schemes. We can see in Fig. 3 that numerical oscillations appear with $(S_{1,1,1}^{N,\Delta t})$ for $u$ and $v$. We can note that they begin also to appear with $(S_{\text{Ref}}^{N,\Delta t'})$ since $\Delta t' = \Delta t > \Delta t$. But no numerical instabilities appear after. The interest of the scheme $(S_{1,1,1}^{N,\Delta t})$ is the computation of the height $h$ since we can see on Fig. 3 that the numerical results obtained with $(S_{1,1,1}^{N,\Delta t})$ are much better than the ones obtained with $(S_{\text{Ref}}^{N,\Delta t'})$.

Now we look for the study of the numerical stability. For this aim, we are interested to obtain an estimate of the amplification of the error on the small scales $\tilde{U}^n_{N_1}$ for one time step for the schemes (4.9) (quasistatic approximation) and (4.15) (explicit scheme with a time step $(l + 1)\Delta t$).

We denote by $\Delta \tilde{U}^n_{N_1}$ the perturbation at time $(n + 1)\Delta t$ for the scheme (4.9). The equation satisfied by $\Delta \tilde{U}^n_{N_1}$ is obtained after linearization of (4.9):

$$\Delta \tilde{U}^n_{N_1} = \Delta \tilde{U}^{n-1}_{N_1} + \Delta \tilde{Q}^n_{N_1} L(\tilde{U}^{n-1}_{N_1}, \Delta \tilde{U}^{n-1}_{N_1}).$$

So the amplification factor for the quasistatic approximation (4.9) is:

$$|\Delta \tilde{U}^n_{N_1}|_{L^2(\mathcal{M})} \leq (1 + C(\tilde{U}^0_{N_1}) \Delta t) |\Delta \tilde{U}^{n-1}_{N_1}|_{L^2(\mathcal{M})},$$

where $C(\tilde{U}^0_{N_1})$ is a scalar constant depending on $\tilde{U}^0_{N_1}$.
Fig. 3. The reference scheme

Fig. 4. Numerical comparison of \((\tilde{S}_{\text{Ref}}^{N,\Delta t})\), \((\tilde{S}_{\text{Ref}}^{N,\Delta t'})\) and \((\tilde{S}_{1,N_1,t}^{N,\Delta t})\) for \(N = 64, N_1 = 32, l = 10, \Delta t = 10\) \(s\) and \(\Delta t' = l\Delta t\). The comparison is done on the computation of the velocity field \(u = (u, v)\) and the height \(h^*\) with \(h = H + h^*\), just before \((\tilde{S}_{\text{Ref}}^{N,\Delta t'})\) stops computing due to numerical instabilities.

For the scheme (4.15) we denote by \(\Delta \tilde{U}_{N_1}^{N,n+1}\) the perturbation at time \((n+1)\Delta t\). The equation satisfied by \(\Delta \tilde{U}_{N_1}^{N,n+1}\) is:

\[
\Delta \tilde{U}_{N_1}^{N,n+1} = \Delta \tilde{U}_{N_1}^{N,n-1} + (l + 1)\Delta t L(\tilde{u}_{N_1}^n, \Delta \tilde{U}_{N_1}^{N,n-1}),
\]

and the amplification factor is:

\[
|\Delta \tilde{U}_{N_1}^{N,n+1}|_{L^2(M)} \leq \left(1 + C(\tilde{u}_{N_1}^n)(l + 1)\Delta t\right)|\Delta \tilde{U}_{N_1}^{N,n-1}|_{L^2(M)}.
\]  

(4.21)

Proposition 3. Comparing (4.20) and (4.21) we observe that the estimation of the amplification factor is smaller for the quasistatic approximation (4.9) than for the explicit scheme (4.15) with a time step \((l + 1)\Delta t\).

To complete the previous study of the numerical stability, we have numerically compared the explicit scheme with a time step \(\Delta t' = l\Delta t\) and the quasistatic approximation over \(l\) time steps with a time step \(\Delta t\), for \(l = 10\), which are nearly equivalent in CPU time since the CPU time is essentially consumed in the computation of the FFTs on the fine level \(N = 64\) rather than on the coarse level \(N_1 = 32\). We have noted that the explicit scheme \((\tilde{S}_{\text{Ref}}^{N,\Delta t'})\) is quickly unstable, as the quasistatic scheme \((\tilde{S}_{1,N_1,t}^{N,\Delta t'})\) is still running. More precisely, the reference scheme \((\tilde{S}_{\text{Ref}}^{N,\Delta t'})\) is stable as we can see on Fig. 3. The reference scheme \((\tilde{S}_{\text{Ref}}^{N,\Delta t'})\) with \(\Delta t' = 5\Delta t\) \((l = 5)\) is also stable as we can see on Fig. 3. But with \(\Delta t' = 10\Delta t\) \((l = 10)\), it is unstable (see numerical oscillations on Fig. 4) just before it stops computing due to numerical instabilities. The scheme \((\tilde{S}_{1,N_1,t}^{N,\Delta t'})\) for \(l = 5\) is stable as we can see on Fig. 3 and for \(l = 10\) it is still stable in opposition with \((\tilde{S}_{\text{Ref}}^{N,\Delta t'})\) which is unstable for \(\Delta t' = 10\Delta t\) (the curves associated with \((\tilde{S}_{\text{Ref}}^{N,\Delta t'})\) and \((\tilde{S}_{1,N_1,t}^{N,\Delta t'})\) are similar, see Fig. 4).

4.2. Multiscale scheme \((S_2)\): explicit/semi-implicit method

With the multiscale scheme \((\tilde{S}_{1,N_1,t}^{N,\Delta t'})\) for \(l = 10\) we have noted that the accuracy is deteriorated, even if it is more stable than the reference scheme \((\tilde{S}_{\text{Ref}}^{N,\Delta t})\). With (3.6) we can remark that the stability constraint from the explicit scheme (3.5) is stronger for the high modes than for the low modes. So we propose a multiscale scheme based on explicit/semi-implicit
schemes. Rather than keeping \( U_{n1}^{N1} \) constant over \( l \) time steps, and to take into account the different dynamics of \( U_{n1} \) and \( U_{n1}^{N1} \), we propose to compute \( U_{n1} \) and \( U_{n1}^{N1} \) with different time schemes and with different time steps.

The quantity \( U_{n1} \) is computed with the explicit time scheme (3.5) and with a time step \( \Delta t \). The semi-implicit scheme retained here is a second order Crank–Nicholson scheme (implicit) for the time integration of the gravity terms, and a second order Leap-Frog scheme (explicit) for the rotation and convective terms. So we obtain, here is a second order Crank–Nicholson scheme (implicit) for the time integration of the gravity terms, and a second order

\[
\begin{align*}
\hat{\omega}_{k}^{n+1} &= \exp(-2\nu_{1}\Delta t |k|^4p)\hat{\omega}_{k}^{n-1} - 2\Delta t \exp(-\nu_{1}\Delta t |k|^4p)(\hat{T}_{\omega,N}(k) - \hat{f}_{\omega,N}(k)). \\
\hat{h}_{k}^{n+1} &= gl\Delta t |k|^2\hat{h}_{k}^{n-1} + g|l\Delta t |k|^2 \exp(-2\nu_{1}\Delta t |k|^4p)\hat{h}_{k}^{n-1} + \exp(-2\nu_{1}\Delta t |k|^4p)\hat{h}_{k}^{n-1} - 2\Delta t \exp(-\nu_{1}\Delta t |k|^4p)(\hat{T}_{h,N}(k) - \hat{f}_{h,N}(k)). \\
\hat{h}_{k}^{n+1} &= -\Delta t \exp(-\nu_{1}\Delta t |k|^4p)(\hat{T}_{h,N}(k) - \hat{f}_{h,N}(k)). \\
\end{align*}
\]

(4.22)

with \( h^* \) defined in (2.2) and \( T_{h^*,N} = \text{div}(h^* u_{n1}) \). The drawback of the semi-implicit scheme is that the increase of the stability is obtained by reducing the speed of the wave propagation (dispersive error). So if we use the semi-implicit scheme to compute all the scales we obtain a better stability but with a dispersive error. Here we intend to apply the semi-implicit scheme only to compute the small scales (high modes), in order to increase the stability but with fewer dispersive error on the large scales which contain most of the kinetic energy and enstrophy (see Williamson and Laprise [39], Dubois et al. [18]).

The computation of the large scales \( U_{n1}^{N+1} \) requires that we know the small scales \( U_{n1}^{N1} \) at the previous time. This is due to the nonlinear terms coupling the large and small scales (see [4.5]). The closure is obtained, as for the scheme \((S_1)\) keeping these nonlinear terms constant at the time \( t_n = n\Delta t \), over the \( l \) time steps \( t_{n+i} = (n+i)\Delta t \), \( i = 1, \ldots, l \) (quasistatic approximation of the nonlinear terms). We obtain the multiscale scheme \((S_{2,N_1,l})\).

In Fig. 5 we have compared \( u \) and \( h \) computed with \((S_{\text{Ref}}^{N_1})\) and \((S_{2,N_1,l})\) for \( N = 64 \), \( N_1 = 32 \) and \( \Delta t = 10 \) s. For the parameter \( l \) we have retained two values for the numerical tests, \( l = 5 \) and \( l = 10 \). Note that the CPU time is nearly the same for \((S_{\text{Ref}}^{N_1})\) and \((S_{2,N_1,l})\), with \( \Delta t' = l\Delta t \). Indeed, the number of FFTs on the fine level, which are the most CPU time

---

**Fig. 5.** Numerical comparison of \((S_{\text{Ref}}^{N_1})\) and \((S_{2,N_1,l})\) for \( N = 64 \), \( N_1 = 32 \), \( l = 5 \) or \( l = 10 \) and \( \Delta t = 10 \) s. The comparison is done on the computation of the velocity field \( u = (u, v) \) and the height \( h^* \) with \( h = H + h^* \).
Fig. 6. Numerical comparison of \((S^N_{\Delta t,1})\), \((S^N_{\Delta t,2})\) and \((S^N_{\Delta t}\text{Impl})\) for \(N = 64, N_1 = 32, l = 10, \Delta t = 10\) s and \(\Delta t' = l\Delta t\). The comparison is done on the computation of the velocity field \(u = (u,v)\) and the height \(h^*\) with \(h = H + h^*\).

expensive, is the same. We can see in Fig. 5 that the accuracy obtained on the velocity field \(u\) is good with \((S^N_{\Delta t,2})\) for \(l = 5\) and \(l = 10\). As for the height \(h\), we can note that the accuracy is good for \(l = 5\), but there is a deterioration on the accuracy for \(l = 10\) since the scheme \((S^N_{\Delta t,1})\) tends to smooth the curve and this increases with \(l\).

In Fig. 6 we compare \((S^N_{\Delta t,2})\) and \((S^N_{\Delta t,1})\) with \((S^N_{\Delta t}\text{Impl})\) for \(N = 64, N_1 = 32, \Delta t = 10\) s, \(l = 10\) and \(\Delta t' = l\Delta t\). We have used the notation \((S^N_{\Delta t,1})\) for the semi-implicit scheme (corresponding to \((S^N_{\Delta t,0,1})\)). For the time step \(\Delta t'\) the semi-implicit scheme is numerically stable, as for the explicit scheme \((S^N_{\Delta t,1})\) is quickly unstable. Note that the CPU time is approximately the same for \((S^N_{\Delta t,1})\) and \((S^N_{\Delta t,2})\). Indeed, with the semi-implicit scheme the nonlinear terms are time discretized in an explicit manner, so there is no nonlinear problem to solve. As for the linear terms, the terms which are time discretized in an implicit manner are the gravity terms and the hyperdissipative operator (2.3). However, the Fourier basis are eigenvectors of the \(\Delta\) and \(\Delta^{2p}\) operators, and since it is an orthogonal basis, the matrix associated is diagonal. So the most CPU time consuming is the computation of the FFTs and the number of FFTs on the fine level \(N\) is the same for \((S^N_{\Delta t,1})\) and \((S^N_{\Delta t,2})\). We can see in Fig. 6 that the numerical results obtained with \((S^N_{\Delta t,1})\) and \((S^N_{\Delta t,2})\) are similar for the velocity field in comparison with \((S^N_{\Delta t,0})\). For the height, the profile of the curve is better suited with \((S^N_{\Delta t,1})\) than with \((S^N_{\Delta t,2})\).

4.3. Multiscale scheme \((S_3):\) averaging method

Here, to take into account the different dynamics of the large and small scales, we intend to replace quasistatic approximation by a diagnostic equation to compute the small scales. We consider time averaged equation for the small scales. Let us denote:

\[
\langle U \rangle_{t, \tau} = \frac{1}{\tau} \int_t^{t+\tau} U(s) \, ds.
\]
For \( t = t_n \) we have:
\[
(U)_{n,\tau} = \frac{1}{\tau} \int_{t_n}^{t_n+\tau} U(s) \, ds.
\]
Noting that \( \frac{d}{dt} (U)_{n,\tau} = \frac{d}{dt} (U)_{n,\tau} \), for \( t = t_n \) we have \( \frac{d}{dt} (U)_{n,\tau} = \frac{d}{dt} (U)_{n,\tau} \) and we can deduce from (4.4) that:
\[
\frac{d}{dt} (U^N_{n_1})_{n,\tau} + v_T \Delta^2 p (U^N_{n_1})_{n,\tau} + (Q^N_{n_1} T^N_{n_1})_{n,\tau} = (Q^N_{n_1} F^N_{n_1})_{n,\tau},
\]
where we have denoted \( T^N = (T_{\omega,H}, T_{\delta,N}, T_{h,N}) \) and \( F^N = (F_{\omega,N}, F_{\delta,N}, F_{h,N}) \). Since the time derivative of \( (U^N_{n_1})_{n,\tau} \) decreases when \( \tau \) increases, we suppose that \( \frac{d}{dt} (U^N_{n_1})_{n,\tau} \simeq 0 \) and we retain to compute \( (U^N_{n_1})_{n,\tau} \) by the following diagnostic equation:
\[
v_T \Delta^2 p (U^N_{n_1})_{n,\tau} + (Q^N_{n_1} T^N_{n_1})_{n,\tau} = (Q^N_{n_1} F^N_{n_1})_{n,\tau}.
\]
We propose to solve Eq. (4.24) with a time step \( \Delta t' = l \Delta t > \Delta t \) and \( \tau = \Delta t' \) to compute the small scales. As for the nonlinear coupling terms between the large and small scales (see (4.5)), the closure is obtained, as previously for \( (S^1) \) and \( (S^2) \), keeping them constant over the \( l \) time steps (quasistatic approximation). We call this multiscale scheme \( (S^{N,\Delta t}_{3,N_1,l}) \).

Now we consider another way to obtain the diagnostic equation (4.24). To simplify the notations let us rewrite:
\[
(U^N_{n_1,avg}(t)) = (U^N_{n_1,l,t,\tau});
\]
we have (see (4.23) and the nonlinear coupling terms (4.5)):
\[
U^N_{n_1,avg}(t) = U^N_{n_1,avg}(t, U_{n_1,avg}(t)).
\]
So, we obtain:
\[
\frac{d}{dt} U^N_{n_1,avg}(t) = \frac{d}{dt} U^N_{n_1,avg}(t, U_{n_1,avg}(t)) + DU^N_{n_1,avg}(t, U_{n_1,avg}(t)) \cdot U'_{n_1,avg}(t),
\]
where \( DU^N_{n_1,avg} \) is the differential of \( U^N_{n_1,avg} \) with respect to \( U_{n_1,avg} \). Eq. (4.23) can be written:
\[
\frac{d}{dt} U^N_{n_1,avg}(t) = -v_T \Delta^2 p U^N_{n_1,avg} + Q^N_{n_1} F_{n,avg} - Q^N_{n_1} (T_{n,avg} - T_{n_1,avg}).
\]
Neglecting the term \( \frac{d}{dt} U^N_{n_1,avg} \) in (4.25) we obtain from (4.26) the diagnostic equation (4.24). This is equivalent to say that the time evolution of the small scales \( U^N_{n_1,avg} \) is essentially due to the time evolution of the large scales \( U_{n_1,avg} \) (slaved law for the small scales in terms of a function of the large scales).

In Fig. 7 we have compared \( u \) and \( h \) computed with \( (S^{Ref}_{3,N_1,l}) \) and \( (S^{N,\Delta t}_{3,N_1,l}) \) for \( N = 64, N_1 = 32 \) and \( \Delta t = 10 \) s. Note that the CPU time is nearly the same for \( (S^{Ref}_{3,N_1,l}) \) and \( (S^{N,\Delta t}_{3,N_1,l}) \), where \( \Delta t' = l \Delta t \), since the most CPU time expensive is the computation of the FFTs on the fine level \( N \) and the number of FFTs on the fine level is the same. We have retained \( l = 5 \) (\( \Delta t' = 5 \Delta t \)) and \( l = 10 \) (\( \Delta t' = 10 \Delta t \)) for the numerical tests. We can see that there is a good approximation of the velocity field \( u \) computed with \( (S^{N,\Delta t}_{3,N_1,l}) \), for \( l = 5 \) and of the height \( h \) computed with \( (S^{N,\Delta t}_{3,N_1,l}) \), for \( l = 5 \) and \( l = 10 \), in comparison with the reference simulation \( (S^{Ref}_{3,N_1}) \).

In Fig. 8, we have represented \( u \) and \( h \) computed with \( (S^{N,\Delta t}_{3,N_1,l}) \) and \( (S^{N,\Delta t'}_{3,N_1,l}) \), for \( N = 64, N_1 = 32, \Delta t = 10 \) s, \( l = 10 \) and \( \Delta t' = l \Delta t \). We recall that the explicit scheme \( (S^{Ref}_{3,N_1,l}) \) is quickly unstable for this value of the time step \( \Delta t' \) and that \( (S^{N,\Delta t}_{3,N_1,l}) \) and \( (S^{N,\Delta t'}_{3,N_1,l}) \) are similar for the CPU time since the number of FFTs evaluated on the fine level is the same for these two schemes. The numerical results obtained with \( (S^{N,\Delta t}_{3,N_1,l}) \) and \( (S^{N,\Delta t'}_{3,N_1,l}) \) are of the same order of accuracy for the velocity field \( u \) in comparison with \( (S^{Ref}_{3,N_1,l}) \). However, the numerical results obtained on the height \( h \) are much better for \( (S^{N,\Delta t}_{3,N_1,l}) \) than for \( (S^{N,\Delta t'}_{3,N_1,l}) \), when compared with the reference scheme \( (S^{Ref}_{3,N_1}) \).

5. New schemes based on a splitting of the operators (fractional step scheme)

Rather than schemes based on a splitting of the scales, as previously considered in Section 4, we now present a scheme based on a splitting of the operators (fractional step scheme). With the splitting of the operators, we can adapt the time integration to the terms considered in the equations.

The nonlinear terms are costly in CPU time. Indeed, we use the FFTs to evaluate the nonlinear terms (product of Fourier series associated with the unknowns) in spectral methods. In two spatial dimensions, a product of two Fourier series,
Fig. 7. Numerical comparison of \((S^N_{\text{Ref}}, \Delta t)\) and \((S^N_{3, N_1, l}, \Delta t)\), for \(N = 64, N_1 = 32, l = 5\) or \(l = 10\) and \(\Delta t = 10\) s. The comparison is done on the computation of the velocity field \(u = (u, v)\) and the height \(h^*\) with \(h = H + h^*\).

truncated to \(N\) modes, required \(O(N^4)\) operations. With the FFTs, this cost is reduced to \(O(N^2 \log_2(N))\) operations since we use the FFTs, which cost \(O(N^2 \log_2(N))\), to switch between physical space and spectral space inversely and, in the physical space, the product of the unknowns cost \(O(N^2)\) operations (product of the values of the unknowns in each of the \(N^2\) points of the grid in the physical space). But the CFL stability constraint, associated with the explicit time integration of the convective nonlinear terms, is not too restrictive (see (3.6)). By comparison, the gravity terms are not costly in CPU time but the stability constraint, associated with the explicit time integration of these terms, is very restrictive (see (3.6)). So, in order to take this into account, we propose to separate the time integration of the nonlinear terms and that of the gravity terms (separation of the operators), and to apply a specific time integration to the nonlinear terms and to the gravity terms.

To consider separately the nonlinear terms and the gravity terms to obtain the solution at time \(t_{n+1} = (n + 1) \Delta t\) from the solution at time \(t_n = n \Delta t\), the time step \(\Delta t\) is decomposed as follows:

- first sub-step to take into account the nonlinear terms (and Coriolis terms) with the explicit Leap-Frog scheme over the time step \(\Delta t\);
- \(n_b\) other sub-steps, \(n_b\) a chosen parameter, \(n_b > 1\) to take into account the gravity terms with the implicit Crank-Nicholson scheme over the time step \(\Delta t\).

So, using \(h = H + h^*\) with \(H = (h) = 10^4\) m (troposphere) and \(h^*\) the height of the free surface around \(H\) (see (2.2)), we consider the following scheme:

- Step one: \(\forall k \in \mathbb{N}_N,$\

   \[
   \begin{align*}
   \omega_{k}^{n+1} &= \exp(-2\nu_T \Delta t |k|^4p)\omega_{k}^{n-1} - 2\Delta t \exp(-\nu_T \Delta t |k|^4p)(\hat{T}_{0,N}^{n}(k) - \hat{F}_{0,N}^{n}(k)), \\
   \delta_{k}^{n+1} &= \exp(-2\nu_T \Delta t |k|^4p)\delta_{k}^{n-1} - 2\Delta t \exp(-\nu_T \Delta t |k|^4p)(\hat{T}_{s,N}^{n}(k) - f \omega_{k}^{n}), \\
   \hat{h}_{k}^{n+1} &= \exp(-2\nu_T \Delta t |k|^4p)\hat{h}_{k}^{n-1} - 2\Delta t \exp(-\nu_T \Delta t |k|^4p)(\hat{T}_{h,N}^{n}(k) - \hat{F}_{h,N}^{n}(k)).
   \end{align*}
   \]
the dispersive error since the influence of the gravity terms, on each sub-step, is reduced.

So we can hope to benefit of the stability properties of the Crank–Nicholson scheme, and to reduce

of the velocity field

in which we have retained only the gravity terms with

time integration for the nonlinear and Coriolis terms, implicit time integration for the gravity terms).

separation of the operators and we use an explicit or implicit time integration according the operators considered (explicit

We obtain the fractional step scheme

Numerical comparison of \( S_{\text{Ref}}^{N,l} \), \( S_{\text{Imp}}^{N,l} \) and \( S_{\text{SN}}^{N,l} \) for \( N = 64 \), \( N_1 = 32 \), \( l = 10 \), \( \Delta t = 10^3 \) and \( \Delta t' = 10^2 \Delta t \). The comparison is done on the computation of the velocity field \( \mathbf{u} = (u, v) \) and the height \( h' \) with \( h = H + h' \).

- Step \( i, i = 2, \ldots, n_b + 1 \) (\( n_b \) sub-steps): \( \forall k \in I_N \),

\[
\begin{align*}
\delta_k^{i} & = \frac{g}{n_b} \Delta t |\mathbf{k}'|^2 \delta_k^{s,i}, \\
\hat{h}_k^{+1} & = \frac{H}{n_b} \Delta t \delta_k^{s,i} - \Delta t \frac{H}{n_b} \exp(-2 \nu_H \Delta t |\mathbf{k}'|^4) \delta_k^{s,i-2}, \\
\delta_k^{s,i-1} & = \frac{\nu_H}{n_b} \Delta t |\mathbf{k}'|^2 \delta_k^{s,i-2} - \Delta t \frac{\nu_H}{n_b} \exp(-2 \nu_H \Delta t |\mathbf{k}'|^4) \delta_k^{s,i-2},
\end{align*}
\]

(5.2)

with \( \delta_k^{s,0} = \hat{h}_k^{+0} \) and \( \hat{h}_k^{+0} = \delta_k^{s,0} \).

- Finally: \( \forall k \in I_N \),

\[
\begin{align*}
\delta_k^{n+1} & = \frac{\tau_{n_b}}{n_b} \delta_k^{s,n_b}, \\
\hat{h}_k^{+1} & = \delta_k^{s,n_b+1}.
\end{align*}
\]

(5.3)

We obtain the fractional step scheme \( S_{\text{SN}, n_b}^{N,l} \) with \( n_b + 1 \) sub-steps. In the first step (5.1), the second (respectively third) equation corresponds to the second (respectively third) equation of (4.22) with \( g = 0 \), \( H = 0 \) and \( l = 1 \). In the \( n_b \) sub-steps (5.2), the second (respectively third) equation corresponds to the second (respectively third) equation of (4.22) with \( l = 1 \) in which we have retained only the gravity terms with \( g/n_b \) and \( H/n_b \) instead of \( g \) and \( H \), reducing the influence of these terms at each sub-step. So we can hope to benefit of the stability properties of the Crank–Nicholson scheme, and to reduce the dispersive error since the influence of the gravity terms, on each sub-step, is reduced.

If we compare the schemes \( S_2 \) and \( S_4 \), in the scheme \( S_2 \) we have a scale separation (explicit scheme for the large scales and semi-implicit scheme for the small scales). In the scheme \( S_4 \) we have not a scale separation, we have a separation of the operators and we use an explicit or implicit time integration according the operators considered (explicit time integration for the nonlinear and Coriolis terms, implicit time integration for the gravity terms).

In Fig. 9 we have compared the velocity field \( \mathbf{u} \) and the height \( h \) computed with \( S_{\text{SN}, n_b}^{N,l} \) and \( S_{\text{SN}, n_b}^{N,l'} \) for \( N = 64 \), \( \Delta t = 10^3 \), \( \Delta t' = n_b \Delta t \) with \( n_b = 10 \). We can see that the scheme \( S_4 \) induces some localized irregularities on the profile of the velocity field \( \mathbf{u} \), essentially on the profile of the \( x \)-component \( u \). No numerical instabilities appear after. As for the \( y \)-component \( v \) of the velocity and for the height \( h \) the curves obtained with \( S_{\text{SN}, n_b}^{N,l} \) and \( S_{\text{SN}, n_b}^{N,l'} \) are similar. Moreover, the
of the velocity field

Fig. 9. Numerical comparison of \((S^{N,\Delta t}_{\text{Ref}}),\ (S^{N,\Delta t}_{\text{Impl}})\) and \((S^{N,\Delta t}_{4,n_b})\) for \(N = 64, n_b = 10, \Delta t = 10\ s\) and \(\Delta t' = n_b \Delta t\). The comparison is done on the computation of the velocity field \(u = (u, v)\) and the height \(h^*\) with \(h = H + h^*\).

Array 1
Comparison of the different schemes \((S_i)\), \(i = 1, \ldots, 4\), \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) and \((S^{N,\Delta t_{\text{Impl}}}_{4,n_b})\) with the reference scheme \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) for \(N = 64, N_1 = 32, \Delta t = 10\ s\) for the test case corresponding to the topography \((2.6)\): time averages of the relative errors for the different global norms \((3.7)\) (lines 1 and 4), \((3.8)\) (line 2), \((3.9)\) (lines 3 and 5) on the velocity field and on the height. We have used the notation \(x e^{-7} = x \times 10^{-7}\). The schemes corresponding to the columns 1, 2, 3 are time CPU equivalent with the scheme \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) of the column 4 and the schemes corresponding to the columns 5, 6, 7 are time CPU equivalent with the scheme \((S^{N,\Delta t_{\text{Impl}}}_{4,n_b})\) of the column 8.

<table>
<thead>
<tr>
<th></th>
<th>((S^{N,\Delta t}_{1,N_1}))</th>
<th>(e^{N,\Delta t_{\text{Ref}}}_{2,N_1}))</th>
<th>((S^{N,\Delta t}_{3,N_1}))</th>
<th>((S^{N,\Delta t}_{4,n_b}))</th>
<th>((S^{N,\Delta t}_{5,\Delta t}))</th>
<th>((S^{N,\Delta t}_{6,\Delta t}))</th>
<th>((S^{N,\Delta t}_{7,\Delta t}))</th>
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<td>1.86e^{-3}</td>
<td>1.23e^{-2}</td>
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<tr>
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<td>6.58e^{-2}</td>
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<tr>
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<td>7.65e^{-2}</td>
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<td>7.79e^{-2}</td>
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<td>1.56e^{-2}</td>
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</tr>
<tr>
<td>Err (L^2_{(h)})</td>
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<td>5.60e^{-4}</td>
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<td>5.09e^{-3}</td>
<td></td>
</tr>
</tbody>
</table>

computational time is less for \((S^{N,\Delta t}_{4,n_b})\) than for \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) since the CPU time for \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) is nearly the same than for \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) (same number of FFTs evaluated on the fine level \(N\), which is the most time-consuming).

In Fig. 9 we have also compared \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) and \((S^{N,\Delta t_{\text{Impl}}}_{4,n_b})\), since the CPU time is nearly the same for these two schemes, with \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\). We have retained \(N = 64, \Delta t' = n_b \Delta t\) with \(n_b = 10\) and \(\Delta t = 10\ s\) for the numerical computation. We recall that \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) is numerically unstable. Comparing with \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\), we can see in Fig. 9 that the numerical results obtained with \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) and \((S^{N,\Delta t_{\text{Impl}}}_{4,n_b})\) have similar accuracy for the velocity field \(u\). However, for the computation of the height \(h\), the numerical results obtained with \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) are better than with \((S^{N,\Delta t_{\text{Impl}}}_{4,n_b})\).

In Array 1 we have compared the different schemes \((S_i)\), \(i = 1, \ldots, 4\), \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) and \((S^{N,\Delta t_{\text{Impl}}}_{4,n_b})\) with the reference scheme \((S^{N,\Delta t_{\text{Ref}}}_{4,n_b})\) for \(N = 64, N_1 = 32, \Delta t = 10\ s\) for the test case corresponding to the topography \((2.6)\). More precisely we have reported the time averages, computed over all the time interval \((80 \times 10^3\) time steps, see Section 3), of the relative errors for the different global norms \((3.7)\) (lines 1 and 4), \((3.8)\) (line 2), \((3.9)\) (lines 3 and 5) on the velocity field and on the height. We have used the notation \(x e^{-7} = x \times 10^{-7}\).
Array 2

Comparison of the different schemes \((S_i), i = 1, \ldots, 4, (\text{SN}_{5,\Delta t})\) and \((\text{SN}_{10,\Delta t})\) with the reference scheme \((S_{\text{Ref}}^{\Delta t})\) for \(N = 64, N_1 = 32, \Delta t = 10\) s for the test case corresponding to the topography \((6.1), (6.2)\): time averages of the relative errors for the different global norms \((3.7)\) (lines 1 and 4), \((3.8)\) (lines 2), \((3.9)\) (lines 3 and 5) on the velocity field and on the height. We have used the notation \(x \rightarrow y = x \times 10^{-y}\). The schemes corresponding to the columns 1, 2, 3 are time CPU equivalent with the scheme \((S_{\text{Ref}}^{\Delta t})\) of the column 4 and the schemes corresponding to the columns 5, 6, 7 are time CPU equivalent with the scheme \((S_{\text{Impl}}^{\Delta t})\) of the column 8.

\[
\begin{array}{cccccccc}
\text{Err } L^2 (u) & 2.51 \times 10^{-2} & 1.10 \times 10^{-2} & 2.51 \times 10^{-2} & 1.43 \times 10^{-3} & 1.21 \times 10^{-2} & 2.54 \times 10^{-2} & 2.41 \times 10^{-3} & 2.49 \times 10^{-3} \\
\text{Err } H^1 (u) & 0.27 & 6.59 \times 10^{-2} & 2.07 & 6.46 \times 10^{-2} & 7.05 \times 10^{-2} & 0.27 & 9.98 \times 10^{-2} & 9.95 \times 10^{-2} \\
\text{Err } L^\infty (u) & 2.80 \times 10^{-2} & 3.26 \times 10^{-2} & 2.80 \times 10^{-2} & 9.71 \times 10^{-3} & 3.53 \times 10^{-2} & 2.78 \times 10^{-2} & 1.65 \times 10^{-2} & 1.61 \times 10^{-2} \\
\text{Err } L^2 (h) & 6.52 \times 10^{-4} & 6.69 \times 10^{-4} & 6.52 \times 10^{-4} & 2.39 \times 10^{-3} & 6.50 \times 10^{-4} & 2.68 \times 10^{-4} & 2.79 \times 10^{-3} & 3.67 \times 10^{-3} \\
\text{Err } L^\infty (h) & 2.00 \times 10^{-3} & 6.03 \times 10^{-3} & 2.00 \times 10^{-3} & 8.72 \times 10^{-4} & 4.76 \times 10^{-3} & 6.52 \times 10^{-3} & 8.67 \times 10^{-4} & 1.72 \times 10^{-3} \\
\end{array}
\]

These relative errors confirm the comparisons shown in Figs. 2–9. Let us remind that in Array 1 the schemes corresponding to the columns 1, 2, 3 are time CPU equivalent with the scheme \((S_{\text{Ref}}^{N,5,\Delta t})\) of the column 4 and the schemes corresponding to the columns 5, 6, 7 are time CPU equivalent with the scheme \((S_{\text{Impl}}^{N,10,\Delta t})\) of the column 8.

6. A second test case corresponding to another complex topography

In this section we consider the following complex topography, periodic in space:

\[
Z_{S,N}(x) = P_N Z_S(x)
\]  
(6.1)

with

\[
Z_S(x) = \frac{\mathcal{H}}{(1 + 4 \sin^2 \pi (x - L_x/2) / L_x)^{3/2}} \frac{(1 + 4 \sin^2 \pi (y - L_y/2) / L_y)^{3/2}}
\]  
(6.2)

where \(x = (x, y) \in \mathcal{M}, \mathcal{M} = (0, L_x) \times (0, L_y)\) (see Section 2). \(\mathcal{H} = 4 \times 10^3\) m, \(N = 64\) and \(P_N\) is the orthogonal projection on the space \(V_N = \text{Span} (\exp (ik \cdot \cdot \cdot ))\), \(k \in 1_N\).

In Fig. 10, we have represented the spectrum of \(Z_{S,N}(x)\) (Fig. 10(a)) and the behavior of \(Z_{S,N}(x)\) in the physical space (Figs. 10(b) and 10(c)).

For the comparison of the numerical results obtained, we have compared the profile of the curves associated with the velocity components \(u, v\) and the height \(h\) through a cross section corresponding to a constant value of \(y = 4 \times 10^6\) m. We have also compared some quantities global in space, such as the energy norm, enstrophy norm and the maximum value (see (3.7)-(3.9)).

The choice of the initial condition is obtained as for the previous topography (2.6) (see Section 3) and the comparisons with the reference simulation are done after 10 days (namely \(80 \times 10^3\) time steps for the explicit reference scheme).

In Fig. 11, we have compared \(u\) and \(h\) computed with the reference scheme \((S_{\text{Ref}}^{\Delta t})\) and with the schemes \((S_{1,N_1}^{\Delta t})\), \((S_{N,5}\Delta t)\) and \((S_{N,10}\Delta t)\) for \(N = 64, N_1 = 32, l = 5\) and \(\Delta t = 10\) s. We recall that the schemes \((S_{1,N_1}^{\Delta t})\), \((S_{N,5}\Delta t)\) and \((S_{N,10}\Delta t)\) are similar for the CPU time. We can see on this figure the interest of the new numerical schemes on the computation of the height \(h\).

In Fig. 12, we have compared \(u\) and \(h\) computed with the explicit reference scheme \((S_{\text{Ref}}^{\Delta t})\) and with the schemes \((S_{N,5}\Delta t)^{\Delta t}, (S_{N,10}\Delta t)^{\Delta t}\) and \((S_{N,10}\Delta t)^{\Delta t}\) for \(N = 64, N_1 = 32, l = 10, n_0 = 10, \Delta t = 10\) s and \(\Delta t' = 1\Delta t\). We recall that the schemes \((S_{N,5}\Delta t)^{\Delta t}, (S_{N,10}\Delta t)^{\Delta t}\) and \((S_{N,10}\Delta t)^{\Delta t}\) are similar for the CPU time and that for this time step \(\Delta t'\) the explicit reference scheme \((S_{\text{Ref}}^{\Delta t})\) is quickly unstable. This is the interest of the new numerical schemes on the computational of the height \(h\) is even more pronounced than in Fig. 11.

In Array 2 we have compared the schemes \((S_i), i = 1, \ldots, 4, (\text{SN}_{5,\Delta t})\) and \((\text{SN}_{10,\Delta t})\) with the reference scheme \((S_{\text{Ref}}^{\Delta t})\) for \(N = 64, N_1 = 32\) and \(\Delta t = 10\) s, computing the time averages (over all the time interval namely \(80 \times 10^3\) time steps) of the relative errors with \((S_{\text{Ref}}^{\Delta t})\) for the different global norms (3.8)-(3.9) and for the topography (6.1), (6.2).

These relative errors confirm the comparisons shown in Figs. 11 and 12. We have used the notation \(xe^{-y} = x \times 10^{-y}\).
Fig. 10. The second complex topography $Z_{S,N}$ considered. Representation in the spectral space (energy spectrum) (a). Representation in the physical space through a cross section with a constant value of $y = 4 \times 10^6$ m (b). Representation in the physical space with a global view (c).

where $x = (x, y) \in \mathcal{M}$, $\mathcal{M} = (0, L_x) \times (0, L_y)$ (see Section 2), $a = 2 \times 10^6$ m, $b = 5 \times 10^3$ m, $\lambda = L_x/10$, $N = 64$ and $P_N$ is the orthogonal projection on the space $V_N = \text{Span}\{\exp(ik' \cdot x)\}$, $k' \in \mathbb{Z}_N$.

In Fig. 13, we have represented the spectrum of $Z_{S,N}(x)$ (Fig. 13(a)) and the behavior of $Z_{S,N}(x)$ in the physical space (Figs. 13(b) and 13(c)).

For the comparison of the numerical results obtained, we have compared the profile of the curves associated with the velocity components $u$, $v$ and the height $h$ through a cross section corresponding to a constant value of $y = 4 \times 10^6$ m. We
Fig. 14. We can note on Fig. 15 that with the scheme this figure that the interest of the new numerical schemes on the computation of the height i.e. \((S_{\text{Ref}}^{N,\Delta t})\) for \(N = 64, N_1 = 32, l = 5, \Delta t = 10s\) and \(\Delta t' = \Delta t\). The comparison is done on the computation of the velocity field \(u = (u, v)\) and the height \(h^*\) with \(h = H + h^*\).

have also compared some quantities global in space, such as the energy norm, enstrophy norm and the maximum value (see (3.7)–(3.9)).

For the initial condition associated with the global model considered (2.1) we have retained \(v = 0\) m{s}^{-1}. As for \(u\) and \(h = H + h^*\) they are derived from the two following equations:

\[
(h^* - Z_{S,N})u = C_1, \quad \frac{1}{2}u^2 + gh = C_2
\]

where \(C_1 = 10\) m²s⁻¹ and \(C_2 = gZ_{S,N}\). We run a large number of time iterations until reaching a statistically steady state, i.e. the averages of global quantities in space are approximately time independent. Then, we use this velocity field obtained to start the comparison between the new proposed schemes and the reference explicit scheme.

In Fig. 14, we have compared \(u\) and \(h\) computed with the reference scheme \((S_{\text{Ref}}^{N,\Delta t})\) and with the schemes \((S_{1,N_1,1}^{N,\Delta t})\), \((S_{2,N_1,1}^{N,\Delta t})\), \((S_{3,N_1,1}^{N,\Delta t})\) and \((S_{4,N_1,1}^{N,\Delta t})\) for \(N = 64, N_1 = 32, l = 5\) and \(\Delta t = 10s\). We recall that the schemes \((S_{1,N_1,1}^{N,\Delta t})\), \((S_{2,N_1,1}^{N,\Delta t})\), \((S_{3,N_1,1}^{N,\Delta t})\) and \((S_{4,N_1,1}^{N,\Delta t})\) are similar for the CPU time. We can see on this figure the interest of the new numerical schemes on the computation of the height \(h\).

In Fig. 15, we have compared \(u\) and \(h\) computed with the explicit reference scheme \((S_{\text{Ref}}^{N,\Delta t})\) and with the schemes \((S_{2,N_1,1}^{N,\Delta t})\), \((S_{3,N_1,1}^{N,\Delta t})\), \((S_{4,N_1,1}^{N,\Delta t})\) and \((S_{\text{Ref}}^{N,\Delta t})\) for \(N = 64, N_1 = 32, l = 10, n_0 = 10, \Delta t = 10s\) and \(\Delta t' = l\Delta t\). We recall that the schemes \((S_{2,N_1,1}^{N,\Delta t})\), \((S_{3,N_1,1}^{N,\Delta t})\), \((S_{4,N_1,1}^{N,\Delta t})\) and \((S_{\text{Ref}}^{N,\Delta t})\) are similar for the CPU time and that for this time step \(\Delta t'\) the explicit reference scheme \((S_{\text{Ref}}^{N,\Delta t})\) is quickly unstable. As it has been the case for the two previous topographies, we can note on this figure that the interest of the new numerical schemes on the computation of the height \(h\) is more pronounced than in Fig. 14. We can note on Fig. 15 that with the scheme \((S_{2,N_1,1}^{N,\Delta t})\) for \(l = 10\) some oscillations appear on the computation of the velocity component \(u\); however no numerical instabilities appear after if the computation is extended on a larger time interval.

In Array 3 we have compared the schemes \((S_i), i = 1, \ldots, 4, (S_{\text{Ref}}^{N,5\Delta t})\) and \((S_{\text{Ref}}^{N,10\Delta t})\) with the reference scheme \((S_{\text{Ref}}^{N,\Delta t})\) for \(N = 64, N_1 = 32\) and \(\Delta t = 10s\), computing the time averages (over all the time interval namely \(80 \times 10^3\) time steps) of the relative errors with \((S_{\text{Ref}}^{N,\Delta t})\) for the different global norms (3.8)–(3.9) and for the topography (7.1), (7.2).

These relative errors confirm the comparisons shown in Figs. 14 and 15. We have used the notation \(xe^{-Y} = x \times 10^{-Y}\).
Fig. 12. Numerical comparison of \( S_{\text{Ref},1}^{N, \Delta t} \) with \( S_{2, N_1}^{N, \Delta t} \), \( S_{3, N_1}^{N, \Delta t} \), \( S_{4, N_1}^{N, \Delta t} \) and \( S_{\text{Impl},1}^{N, \Delta t} \) for \( N = 64, N_1 = 32, \Delta t = 10 \) s and \( \Delta t' = 1 \Delta t \). The comparison is done on the computation of the velocity field \( u = (u, v) \) and the height \( h' \) with \( h = H + h' \).

Array 3
Comparison of the different schemes \((S_i)\), \(i = 1, \ldots, 4\), \(S_{\text{Ref},1}^{N, \Delta t} \) and \( S_{\text{Impl},1}^{N, \Delta t} \) with the reference scheme \( S_{\text{Ref},1}^{N, \Delta t} \) for \( N = 64, N_1 = 32, \Delta t = 10 \) s for the test case corresponding to the topography \((7.1), (7.2)\); time averages of the relative errors for the different global norms \((3.7)\) (lines 1 and 4), \((3.8)\) (line 2), \((3.9)\) (lines 3 and 5) on the velocity field and on the height. We have used the notation \( e^{x} = x \times 10^{-2} \). The schemes corresponding to the columns 1, 2, 3 are time CPU equivalent with the scheme \( S_{\text{Impl},1}^{N, \Delta t} \) of the column 4 and the schemes corresponding to the columns 5, 6, 7 are time CPU equivalent with the scheme \( S_{\text{Impl},1}^{N, \Delta t} \) of the column 8.

<table>
<thead>
<tr>
<th>( S_{\text{Ref},1}^{N, \Delta t} )</th>
<th>( S_{2, N_1}^{N, \Delta t} )</th>
<th>( S_{3, N_1}^{N, \Delta t} )</th>
<th>( S_{4, N_1}^{N, \Delta t} )</th>
<th>( S_{\text{Impl},1}^{N, \Delta t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L^2 ) (u)</td>
<td>1.17e-2</td>
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<td>1.83e-3</td>
<td>1.50e-3</td>
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<tr>
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<td>0.24</td>
<td>8.67e-2</td>
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</tr>
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<td>1.88e-2</td>
<td>3.34e-2</td>
<td>9.60e-4</td>
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<tr>
<td>( L^2 ) (h)</td>
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<td>1.30e-3</td>
<td>1.31e-3</td>
<td>1.34e-3</td>
</tr>
<tr>
<td>( L^\infty ) (h)</td>
<td>2.39e-2</td>
<td>2.35e-2</td>
<td>1.98e-2</td>
<td>1.64e-3</td>
</tr>
</tbody>
</table>

8. Conclusion and future work

We have presented in this work several new multiscale/fractional step schemes to compute the numerical solution of the rotating shallow water equations \((2.5)\), with complex varying topography. These schemes are based on a scale separation of the complex topography (multiscale methods), or an operator separation (fractional step methods). The objective is to obtain numerical schemes giving results close to those obtained using only one uniform fine grid \( N \times N \) and a time step \( \Delta t \), but with a CPU time close to the CPU time needed when using only one coarse grid \( N_1 \times N_1 \), \( N_1 < N \) and/or a time step \( \Delta t' > \Delta t \).

We have presented the numerical results obtained with the proposed schemes \((S_i)\), \(i = 1, \ldots, 4\). Comparison is done with the reference simulation \( S_{\text{Ref},1}^{N, \Delta t} \) (explicit scheme) and with \( S_{\text{Impl},1}^{N, \Delta t} \) (semi-implicit scheme). The numerical results obtained show that the schemes proposed reduce the CPU time, increase the numerical stability and preserve some numerical precision on the computation of the velocity field \( u \) and of the height \( h \).

In Arrays 1, 2 and 3 we can see that for the three topographies \((2.6), (6.1) \) and \((6.2), (7.1) \) and \((7.2)\), considered for the numerical simulations, the new proposed schemes \((S_i)\), \(i = 1, \ldots, 4\) give, in comparison with the reference scheme \( S_{\text{Ref},1}^{N, \Delta t} \), better approximations for the height \( h \) of the flow than the classical schemes \( S_{\text{Ref},1}^{N, \Delta t} \) and \( S_{\text{Impl},1}^{N, \Delta t} \), for CPU times equivalent, and this is especially true when the time step is increased.
Fig. 13. The third complex topography $Z_{S,N}$ considered. Representation in the spectral space (energy spectrum) (a). Representation in the physical space through a cross section with a constant value of $y = 4 \times 10^6$ m (b). Representation in the physical space with a global view (c).

More precisely, considering the first topography (2.6) (see Array 1) we have:

- for a time CPU equivalent with the reference scheme using a time step $5\Delta t$, for the height $h$ the schemes $(S_1)$, $(S_2)$ and $(S_3)$ are better than $(S_{N,5\Delta t})$ in the $L^2$ norm and the schemes $(S_1)$, $(S_2)$ are better than $(S_{N,5\Delta t})$ in the $L^\infty$ norm. As for the velocity field $u$ the schemes $(S_1)$ and $(S_2)$ are better than $(S_{N,5\Delta t})$ in the $H^1$ norm.
Fig. 14. Numerical comparison of \( S_{\text{Ref}}^{N,\Delta t} \) with \( S_{1,\text{Ref},i}^{N,\Delta t} \), \( S_{1,\text{Impl},i}^{N,\Delta t} \) and \( S_{\text{Ref}}^{N,5\Delta t} \) for \( N = 64, N_1 = 32, l = 5, \Delta t = 10 \text{s and } \Delta t' = 1\Delta t \). The comparison is done on the computation of the velocity field \( u = (u, v) \) and the height \( h^* \) with \( h = H + h^* \).

- for a time CPU equivalent with the reference scheme using a time step \( 10\Delta t \), for the height \( h \) the schemes \((S_2), (S_3)\) and \((S_4)\) are better than \( (S_{\text{Ref}}^{N,10\Delta t}) \) in the \( L^2 \) and \( L^\infty \) norms. As for the velocity field \( u \), the scheme \((S_4)\) is better than \( (S_{\text{Impl}}^{N,10\Delta t}) \) in the \( L^2 \) and \( L^\infty \) norms, the scheme \((S_2)\) is better than \( (S_{\text{Impl}}^{N,10\Delta t}) \) in the \( H^1 \) norm and the scheme \((S_3)\) is better than \( (S_{\text{Impl}}^{N,10\Delta t}) \) for the \( L^\infty \) norm.

Now, considering the second topography \((6.1), (6.2)\) (see Array 2) we have:

- for a time CPU equivalent with the reference scheme using a time step \( 5\Delta t \), for the height \( h \) the schemes \((S_1), (S_2)\) and \((S_3)\) are better than \( (S_{\text{Ref}}^{N,5\Delta t}) \) in the \( L^2 \) norm. As for the velocity field \( u \), the scheme \((S_2)\) is similar with \( (S_{\text{Ref}}^{N,5\Delta t}) \) in the \( H^1 \) norm.
- for a time CPU equivalent with the reference scheme using a time step \( 10\Delta t \), for the height \( h \) the schemes \((S_2), (S_3)\) and \((S_4)\) are better than \( (S_{\text{Impl}}^{N,10\Delta t}) \) in the \( L^2 \) norm and the scheme \((S_4)\) is better than \( (S_{\text{Impl}}^{N,10\Delta t}) \) in the \( L^\infty \) norm. As for the velocity field \( u \), in the \( L^2 \) norm the scheme \((S_4)\) is better than \( (S_{\text{Impl}}^{N,10\Delta t}) \) and in the \( H^1 \) norm the scheme \((S_2)\) is better than \( (S_{\text{Impl}}^{N,10\Delta t}) \).

Finally for the third topography \((7.1), (7.2)\) (see Array 3) we have:

- for a time CPU equivalent with the reference scheme using a time step \( 5\Delta t \), for the height \( h \) in the \( L^2 \) norm the schemes \((S_1), (S_2)\) and \((S_3)\) are better than \( (S_{\text{Ref}}^{N,5\Delta t}) \). As for the velocity field \( u \), in the \( H^1 \) norm the scheme \((S_3)\) is better than \( (S_{\text{Ref}}^{N,5\Delta t}) \).
- for a time CPU equivalent with the reference scheme using a time step \( 10\Delta t \), for the height \( h \) in the \( L^2 \) and \( L^\infty \) norms the schemes \((S_2), (S_3)\) and \((S_4)\) are better than \( (S_{\text{Impl}}^{N,10\Delta t}) \). As for the velocity field \( u \), in the \( L^2 \) norm the schemes \((S_3)\) and \((S_4)\) are better than \( (S_{\text{Impl}}^{N,10\Delta t}) \).

Another planetary model is the two dimensional shallow water problem on the sphere (spherical coordinates). In this case, an additional problem is the use of nonuniform meshes (finer near the poles, see Williamson and Laprise [39], for
Fig. 15. Numerical comparison of $(SN, \Delta t_{Ref})$ with $(SN, \Delta t_1, N_1, l)$, $(SN, \Delta t_2, N_1, l)$, $(SN, \Delta t_3, nb)$ and $(SN, \Delta t'_3, \text{Impl})$ for $N = 64$, $N_1 = 32$, $l = 10$, $n_b = 10$, $\Delta t = 10$ s and $\Delta t' = \Delta t$. The comparison is done on the computation of the velocity field $u = (u, v)$ and the height $h^*$ with $h = H + h^*$.

example). So, the presence of the poles implies a more restrictive stability constraint over all the sphere. To overcome this, we can use spectral methods with spherical harmonic basis. For more details, see Orszag [25,26], Williamson et al. [38], Williamson and Laprise [39], Durrant [19]. We intend to adapt the new multiscale and fractional step schemes previously described to this case of a spherical geometry with complex topography.

We shall also consider the case of the system of Eqs. (2.5) on a rectangular domain $\mathcal{M}$, with Dirichlet boundary conditions. This problem is considered as a Limited Area Model (LAM) in meteorology, also called Regional Problem (see Sundström and Elvius [35], Coiffier [14]). Such a model allows, in meteorology, a prediction for the small scales on a short period, in contrast to a planetary model, which is used for the prediction of the large scales on a long period. Indeed, in the case of a limited area model, the mesh is finer than for a planetary model. For a LAM, it is necessary to specify the values of the velocity field on the boundary of the limited area domain (Dirichlet boundary conditions), at each time step. These values, on the boundary, can be obtained using a model for the large scales (for example a planetary model). For more details, see, for example, Sundström and Elvius [35], Coiffier [14]. We intend to adapt the previous multiscale and fractional step schemes, to the case of a limited area model. For such a problem, the numerical approximation is often obtained using a finite difference or a finite volume method. To obtain large and small scales decomposition in the finite difference case or in the finite volume case, we can use the Incremental Unknowns (IU) method. For more details on the incremental unknowns see, for example, Chen and Temam [13], Chen et al. [12], Chehab and Miranville [11], Chehab and Costa [10], Faure [20] and the references therein. For our approach, a spectral element method seems better suited (see Patera [28], Baer et al. [1,2]) for which scale separation is more natural.

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