THE NONLINEAR 2D SUBCRITICAL INVISCID SHALLOW WATER EQUATIONS WITH PERIODICITY IN ONE DIRECTION

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Abstract. In continuation with earlier works on the shallow water equations in a rectangle [10, 11], we investigate in this article the fully inviscid nonlinear shallow water equations in space dimension two in a rectangle \((0,1)_x \times (0,1)_y\).

We address in this article the subcritical case, corresponding to the condition (3) below. Assuming space periodicity in the \(y\)-direction, we propose the boundary conditions for the \(x\)-direction which are suited for the subcritical case and develop, for this problem, results of existence, uniqueness and regularity of solutions locally in time for the corresponding initial and boundary value problem.

1. Introduction. Motivated by the study of the well-posedness of the primitive equations of the atmosphere and the ocean, we consider in this article the inviscid fully nonlinear shallow water equations (SWE)

\[
\begin{align*}
    u_t + uu_x + vu_y + g\phi_x - fv &= 0, \\
    v_t + uv_x + vv_y + g\phi_y + fu &= 0, \\
    \phi_t + u\phi_x + v\phi_y + \phi(u_x + v_y) &= 0;
\end{align*}
\]

(1)

here \(U = (u, v, \phi)^t\), \(t \in (0, T)\), \((x, y) \in \Omega := (0, 1)_x \times \mathbb{T}_y\) where \(\mathbb{T} = \mathbb{T}_y\) is the 1-torus, with the endpoints of the interval \([0, 1]\) identified; \(u\) and \(v\) are the two horizontal components of the velocity, \(\phi\) is the height of the water, and \(g\) is the gravitational acceleration, \(f\) is the Coriolis parameter. The first and second equations (1) are derived from the conservation of horizontal momentum, and the third one expresses the conservation of mass.

The relation of the shallow water equations with the primitive equations of the atmosphere and the ocean is explained in e.g. [11, 20] in which it appears that the system (1) is equivalent to one mode of a certain vertical expansion of the primitive equations.

We want to associate with (1) suitable boundary (and initial) conditions leading to a well-posed problem. This system is generally hyperbolic and although the theory of multi-dimensional boundary value problems in smooth domains is well understood (see e.g. [2]), the theory in nonsmooth domains such as \(\Omega\) is much less developed and much less is known for a problem like (1); see however the articles

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which study the boundary value problem for linear hyperbolic equations in domains with corners; the author showed that various singularities can occur near the corners of the domain for certain boundary conditions. More precisely the author considered in [17] the two dimensional shallow water equations linearized around a constant flow and provided in Theorem 2.6 of this article situations called \((\tilde{a})\) which produce a lack of uniqueness and \((\tilde{b})\) which produce a lack of existence of solution.

In the linear case, a fairly general theory of existence and uniqueness of solutions for (a linearization of) (1) has been developed in [11], and more general linear hyperbolic problems in e.g. a rectangle are considered in [12]. In [10] we have considered the fully nonlinear inviscid shallow water equations in the case corresponding to a supercritical flow. In this article we consider a subcritical flow in the domain \((0,1)_x \times (0,1)_y\) with space periodicity in the \(y\)-direction. We then propose boundary conditions in the \(x\)-direction which make the initial and boundary problem well-posed. We derive indeed results of existence, uniqueness and regularity of solutions, locally in time, for the corresponding initial and boundary value problem.

This article is organized as follows. After this introductory section, we first study in Section 2 the boundary value problem for the linearized shallow water system in the \(L^2\) space and then extend it to the Sobolev spaces \(H^m\). We are concerned here with the domain \(Q = \Omega \times \mathbb{R}_t\), and equation (1) are linearized around a flow \(\ddot{U} = (\ddot{u}, \ddot{v}, \ddot{\phi})\) which is subcritical in the sense of (3) below, the regularity of \(\ddot{U}\) being specified as needed. In Section 3, we consider the linear SWE with homogeneous initial condition, that is in \(\Omega \times (0,T)\). Finally, Section 4 is devoted to proving the local well-posedness for the fully nonlinear SWE in the subcritical case in \(\Omega \times (0,T)\) by a standard iteration scheme. In the Appendices A and B, we collect some useful theorems about the Sobolev spaces on the torus, and several classical estimates about functions in the Sobolev spaces. In the Appendix C, we prove a trace result which we used in the article.

2. Boundary value problem. In this section, we aim to study the well-posedness of the following boundary value problem in the subcritical case (see (3) below):

\[
L_0 U := U_t + \mathcal{E}_1(\ddot{U}) \partial_x U + \mathcal{E}_2(\ddot{U}) \partial_y U + \ell(U) = F,
\]

(2)

associated with suitable boundary conditions specified below. Here \((x,y,t) \in Q := (0,1)_x \times T_y \times \mathbb{R}_t\), \(\ell(U) = (-fv, fu, 0)^t\), and

\[
\mathcal{E}_1(\ddot{U}) = \begin{pmatrix} \ddot{u} & 0 & g \\ 0 & \ddot{u} & 0 \\ \ddot{\phi} & 0 & \ddot{u} \end{pmatrix}, \quad \mathcal{E}_2(\ddot{U}) = \begin{pmatrix} \ddot{v} & 0 & 0 \\ 0 & \ddot{v} & g \\ 0 & \ddot{\phi} & \ddot{v} \end{pmatrix}.
\]

The regularity of \(\ddot{U}\) is specified below. Note that \(\mathcal{E}_1^1, \mathcal{E}_2\) admit a symmetrizer \(S_0 = \text{diag}(1,1,g/\ddot{\phi})\), which means that \(S_0\mathcal{E}_1,S_0\mathcal{E}_2\) are both symmetric. In the following, we assume for simplicity that \(\ddot{u}, \ddot{v}\) are positive. The case where \(\ddot{u}\) and, or \(\ddot{v}\) are negative can be treated in a similar manner; we do not study the non-generic case when \(\ddot{u}\) or \(\ddot{v}\) vanishes. By its physical meaning (height), \(\ddot{\phi}\) is positive and we do not study the case where \(\ddot{\phi} = 0\). As we indicated before, we only study the

\[\text{We sometimes write } \mathcal{E}_1 = \mathcal{E}_1(\ddot{U}) \text{ for short, etc.}\]
subcritical case; we thus assume that \( \hat{U} \) satisfies the enhanced subcritical condition:

\[
\begin{align*}
  c_0 \leq \hat{U} & \leq c_1, \\
  \hat{u}^2 - g\hat{\phi} & \leq -c_2^2,
\end{align*}
\]

where \( c_0, c_1, c_2 \) are positive constants. Note that we do not impose any relation on \( \hat{v} \) and \( \hat{\phi} \) since the \( y \)-direction is periodic (no boundary on the 1-torus \( \mathbb{T} \)).

Our agenda in this section is as follows. We first derive suitable boundary conditions for (2) in Subsection 2.1 and then address the issue of the \( L^2 \) well-posedness of (2) in Subsection 2.2 based on regularization and variational methods. Subsections 2.3 and 2.5 are devoted to establishing the \( L^2 \) and \( H^m \) a priori estimates. Subsection 2.4 aims to show that the solutions of (2) enjoy the same regularity as the boundary conditions and sourcing terms, and finally Subsection 2.6 concerns the existence and uniqueness of the solutions for (2) in \( H^m \).

2.1. Boundary conditions. Under the subcritical condition (3), the matrix \( E_1 \) has two positive eigenvalues \( \hat{u} \) and \( \hat{u} + \sqrt{g\hat{\phi}} \) and one negative eigenvalue \( \hat{u} - \sqrt{g\hat{\phi}} \), and we find that

\[
P E_1 P^{-1} = \text{diag}(\hat{u} + \sqrt{g\hat{\phi}}, \hat{u}, \hat{u} - \sqrt{g\hat{\phi}}),
\]

where

\[
P = \begin{pmatrix} 1 & 0 & \sqrt{g/\hat{\phi}} \\ 0 & 1 & 0 \\ 1 & 0 & -\sqrt{g/\hat{\phi}} \end{pmatrix}.
\]

We then set

\[
B_0(\hat{U}) = \begin{pmatrix} 1 & 0 & \sqrt{g/\hat{\phi}} \\ 0 & 1 & 0 \\ 1 & 0 & -\sqrt{g/\hat{\phi}} \end{pmatrix}, \quad B_1(\hat{U}) = \begin{pmatrix} 1 & 0 & \sqrt{g/\hat{\phi}} \\ 0 & 1 & 0 \\ 1 & 0 & -\sqrt{g/\hat{\phi}} \end{pmatrix}.
\]

From the general Kreiss-Lopatinski\' theory for hyperbolic boundary value problems (see e.g. [13, 15] or Chapter 4 in [2]), in order to solve (2), it is necessary and sufficient to specify the boundary conditions for \( B_0(\hat{U})U \) on \( x = 0 \) and \( B_1(\hat{U})U \) on \( x = 1 \). We thus choose to specify the boundary conditions \( G = (g_1, g_2, g_3)^t \) for \( U \) such that:

\[
\begin{align*}
  B_0(\hat{U})U &= (g_1, g_2, 0)^t, \text{ at } x = 0, \\
  B_1(\hat{U})U &= g_3, \text{ at } x = 1.
\end{align*}
\]

We sometimes use the following short notation to stand for (5):

\[
B(\hat{U})U = G, \text{ on } \partial Q.
\]

It is useful to note that the boundary conditions (5) are strictly dissipative under the subcritical condition (3), meaning that there exists \( \epsilon_0 > 0 \) and \( C_0 > 0 \) such that

\[
\begin{align*}
  (S_0(\hat{U})E_1(\hat{U})U, U) &\geq \epsilon_0|U|^2 - C_0|B_1(\hat{U})U|^2, \forall U \in \mathbb{R}^3 \text{ at } x = 1, \\
  (S_0(\hat{U})E_1(\hat{U})U, U) &\leq -\epsilon_0|U|^2 + C_0|B_0(\hat{U})U|^2, \forall U \in \mathbb{R}^3 \text{ at } x = 0,
\end{align*}
\]

where \((\cdot, \cdot)\) (resp. \(|\cdot|\)) denotes the standard inner product (resp. norm) on \( \mathbb{R}^3 \). The relations (7) can be verified by direct computations where \( \hat{U} \) and \( U \) are sufficiently regular, and the constants \( \epsilon_0 \) and \( C_0 \) only depend on \( c_0, c_1, c_2, g \) but are independent of \( \hat{U} \).
Since we have to assign boundary conditions on $\partial \mathcal{Q}$, we first need to make sure that the desired traces at the boundary make sense. We thus consider the space:
\[
\mathcal{X}(\mathcal{Q}) = \{ U \in L^2(\mathcal{Q}) : U_t + \mathcal{E}_1(\tilde{U}) \partial_x U + \mathcal{E}_2(\tilde{U}) \partial_y U \in L^2(\mathcal{Q}) \},
\]
endowed with its natural Hilbert norm
\[
\| U \|_{\mathcal{X}(\mathcal{Q})} = (\| U \|^2_{L^2(\mathcal{Q})} + \| U_t + \mathcal{E}_1(\tilde{U}) \partial_x U + \mathcal{E}_2(\tilde{U}) \partial_y U \|^2_{L^2(\mathcal{Q})})^{1/2}.
\]
Notice that we have written $\| U \|_{L^2(\mathcal{Q})}$ instead of $\| U \|_{L^2(\mathcal{Q})}^2$. Here and in the future as well, we will not distinguish the notations for vector and scalar function spaces whenever they are self-evident from the context. Then we have the following trace result.

**Proposition 2.1.** Assume that $\tilde{U}$ belongs to $W^{1,\infty}(\mathcal{Q})$ and satisfies the subcritical condition (3). If $U$ belongs to $\mathcal{X}(\mathcal{Q})$, then the traces of $U$ are defined at $x = 0, 1$, and the traces $U|_{x=0,1}$ belong to $H^{-1}(T \times \mathbb{R})$. Furthermore the trace operators are linear continuous in the corresponding spaces, e.g. $U \in \mathcal{X}(\mathcal{Q}) \to U|_{x=0}$ is continuous from $\mathcal{X}(\mathcal{Q})$ into $H^{-1}(T \times \mathbb{R})$.

**Proof.** Noting that $U_t, U_y$ belong to $L^2_2(0,1; H^{-1}(T \times \mathbb{R}))$ and the matrix $\mathcal{E}_1(\tilde{U})$ is non-singular, we then infer from $U_t + \mathcal{E}_1(\tilde{U}) \partial_x U + \mathcal{E}_2(\tilde{U}) \partial_y U \in L^2(\mathcal{Q})$ that $U_x \in L^2_2(0,1; H^{-1}(T \times \mathbb{R}))$, which together with $U \in L^2(\mathcal{Q})$, shows that $U \in C([0,1]; H^{-1}(T \times \mathbb{R}))$. Hence the traces on $x = 0, 1$ are well-defined. The continuity of the corresponding mappings is easy. □

### 2.2. $L^2$ well-posedness.
Classically (see e.g. [2, 18]), we need an energy estimate weighted in time, and we thus introduce the new unknown $\tilde{U}_\gamma := e^{-\gamma t}U(x,y,t)$ and new data $\tilde{F}_\gamma, \tilde{G}_\gamma$ defined in the same way. We observe that (2) and (5) are equivalent to
\[
\begin{cases}
L_{\gamma,\tilde{U}} \tilde{U}_\gamma = \tilde{F}_\gamma, \text{ in } \mathcal{Q}, \\
B(\tilde{U}) \tilde{U}_\gamma = \tilde{G}_\gamma, \text{ on } \partial \mathcal{Q},
\end{cases}
\]
where
\[
L_{\gamma,\tilde{U}} := L_{\tilde{U}} + \gamma = \partial_t + \mathcal{E}_1(\tilde{U}) \partial_x + \mathcal{E}_2(\tilde{U}) \partial_y + \ell + \gamma.
\]
Note that there is no initial conditions since we are considering a boundary value problem with $t \in \mathbb{R}$. In order to show the $L^2$ well-posedness, we introduce a suitable change of variable which makes the boundary conditions in (8) simpler. We set $\Xi := P \tilde{U}_\gamma$, where $P$ is defined by (4). Multiplying both sides of (8) by $P$ and using the new variable $\Xi$, we obtain
\[
\Xi_t + \mathcal{E}_1^1(\tilde{U}) \Xi_x + \mathcal{E}_2^1(\tilde{U}) \Xi_y + (PP_{t}^{-1} + PP_{x}^{-1} + PP_{y}^{-1} + \ell + \gamma) \Xi = P\tilde{F}_\gamma,
\]
where
\[
\mathcal{E}_1^1(\tilde{U}) = \begin{pmatrix}
\hat{u} + \sqrt{g\phi} & 0 & 0 \\
0 & \hat{u} & 0 \\
0 & 0 & \hat{u} - \sqrt{g\phi}
\end{pmatrix},
\mathcal{E}_2^1(\tilde{U}) = \begin{pmatrix}
\hat{v} & \sqrt{g\phi} & 0 \\
\sqrt{g\phi}/2 & \hat{v} & -\sqrt{g\phi}/2 \\
0 & -\sqrt{g\phi}/2 & \hat{v}
\end{pmatrix}.
\]
We first consider the homogenous boundary conditions, that is we associate to (9) the following boundary conditions in the $\Xi$ variable (see (5)):
\[
\begin{cases}
\xi = \eta = 0, \text{ at } x = 0, \\
\xi = 0, \text{ at } x = 1.
\end{cases}
\]
To solve (9) and (10), noticing that $\mathcal{E}_1^1(\bar{U})$ and $\mathcal{E}_2^1(\bar{U})$ admit a symmetrizer $S^1 = \text{diag}(1, 2, 1)$, we introduce the following regularized problem

$$
-\epsilon \Delta \Xi^\epsilon + S^1 \Xi^\epsilon_x + S^1 \mathcal{E}_1^1(\bar{U}) \Xi^\epsilon_y + S^1 \mathcal{E}_2^1(\bar{U}) \Xi^\epsilon_y + S^1 (PP_1^{-1} + PE_1(\bar{U})P_2^{-1} + PE_2(\bar{U})P_3^{-1} + \ell + \gamma) \Xi^\epsilon = S^1 P \bar{F}_\gamma,
$$

(11)

endowed with the boundary conditions

$$
\begin{cases}
\xi^\epsilon = \eta^\epsilon = \xi_y^\epsilon = 0, & \text{at } x = 0, \\
\xi_x^\epsilon = \eta_x^\epsilon = \xi_y^\epsilon = 0, & \text{at } x = 1.
\end{cases}
$$

(12)

We also introduce the Hilbert space

$$
V = \{ \Xi \in H^1(\Omega) : \xi = \eta = 0 \text{ at } x = 0; \zeta = 0 \text{ at } x = 1 \}.
$$

(13)

Note that $\|\nabla \Xi\|_{L^2}$ is a natural norm on $V$ thanks to the Poincaré inequality. We now define the linear form $L(\Xi) = \langle S^1 P \bar{F}_\gamma, \Xi \rangle$ and the bilinear form $a$ on $V$:

$$
a(\Xi, \Xi) = \epsilon \langle \nabla \Xi, \nabla \Xi \rangle + \langle S^1 \Xi, S^1 \mathcal{E}_1^1(\bar{U}) \Xi_x + S^1 \mathcal{E}_2^1(\bar{U}) \Xi_y, \Xi \rangle + \langle S^1 (PP_1^{-1} + PE_1(\bar{U})P_2^{-1} + PE_2(\bar{U})P_3^{-1} + \ell + \gamma) \Xi, \Xi \rangle.
$$

(14)

It is not hard to check that $L$ and $a$ are continuous on $V$, and $a$ is coercive for $\gamma$ large enough. Hence by the Lax-Milgram theorem, there exists a unique $\Xi^\epsilon \in V$ such that

$$
a(\Xi^\epsilon, \Xi) = L(\Xi), \quad \forall \Xi \in V.
$$

(15)

Choosing $\Xi \in \mathcal{D}(\Omega)$ in (15) shows that $\Xi^\epsilon$ satisfies (11) in the sense of distributions.

We hence see that $\Delta \Xi^\epsilon$ belongs to $L^2(\Omega)$ and then find that the traces $\Xi^\epsilon|_{x=0,1}$ are well-defined by applying Theorem 1.2 in [25] with $u = \nabla \Xi^\epsilon$. We now choose $\Xi \in C^\infty(\Omega) \cap V$ in (15), integrate by parts, and utilize (11); we arrive at

$$
\epsilon \int_{T \times R} \Xi^\epsilon_x \cdot \Xi^\epsilon|_{x=0}^1 dt dy = 0, \quad \forall \Xi \in C^\infty(\Omega) \cap V,
$$

(16)

which implies that $\xi_y^\epsilon = 0$ at $x = 0$ and $\xi_x^\epsilon = \eta_x^\epsilon = 0$ at $x = 1$. In conclusion, we have

\begin{lemma}
Assume that $\bar{U}$ belongs to $W^{1,\infty}(\Omega)$ and satisfies the subcritical condition (3). Then there exists $\gamma_0 > 0$ only depending on the $W^{1,\infty}(\Omega)$-norm of $\bar{U}$ such that: for all $\gamma > \gamma_0$, and for $F_\gamma \in L^2(\Omega)$, there exists a unique weak solution $\Xi^\epsilon \in V$ of (11)-(12), and furthermore, the $\Xi^\epsilon$ satisfy the following estimates independent of $\epsilon$:

$$
\Xi^\epsilon, \sqrt{\epsilon} \nabla \Xi^\epsilon \text{ are bounded in } L^2(\Omega).
$$

(17)
\end{lemma}

\begin{proof}
We only need to show the estimates (17), which can be obtained by letting $\Xi = \Xi^\epsilon$ in (15).
\end{proof}

The estimate in (17) show that there exists a subsequence of $\Xi^\epsilon$, still denoted by $\Xi^\epsilon$, which converges to some $\Xi$ in $L^2(\Omega)$, and the limit $\Xi$ (at least weakly) solves (9). It remains to verify that $\Xi$ satisfies the boundary conditions (10). Using the estimates (17), (11) gives

$$
-\epsilon \Xi^\epsilon_{xx} + S^1 \mathcal{E}_1^1(\bar{U}) \Xi^\epsilon_x \text{ is bounded in } L^2_x(0,1;H^{-1}(\mathbb{T} \times \mathbb{R})) \text{ independently of } \epsilon; \quad \text{and (18)}
$$

since $S^1 \mathcal{E}_1^1(\bar{U})$ is diagonal, applying Lemma C.1 or Remark 9 with $p = 2$ and $X = H^{-1}(\mathbb{T} \times \mathbb{R})$ to each component of $\Xi^\epsilon$ yields that the $\Xi^\epsilon_x$ are bounded in $L^2_x(0,1;H^{-1}(\mathbb{T} \times \mathbb{R}))$ independently of $\epsilon$ and that the corresponding traces converge,
i.e. $\xi = \eta = 0$ on $x = 0$, $\zeta = 0$ on $x = 1$. Therefore, we find that the limit $\Xi$ satisfies the boundary conditions \((10)\). Transforming back to the original variable $\tilde{U}_\gamma$, we obtain that there exists a solution $\tilde{U}_\gamma \in L^2(Q)$ satisfying \((8)\) with the homogeneous boundary conditions (i.e. $\tilde{G}_\gamma = 0$). Hence, we have

**Proposition 2.3.** Assume that $\tilde{U}$ belongs to $W^{1,\infty}(Q)$ and satisfies the subcritical condition \((3)\). Then there exists $\gamma_0 > 0$ only depending on the $W^{1,\infty}(Q)$-norm of $\tilde{U}$ such that: for all $\gamma > \gamma_0$, for all $F \in \epsilon^t L^2(Q)$, with $G = 0$, there exists a solution $U \in \epsilon^t L^2(Q)$ satisfying \((2)\) and \((5)\).

For the case of non-homogeneous boundary conditions, we assume that the boundary data $G \in \epsilon^t L^2(T \times \mathbb{R})$ is inferred from a function $U^g = (u^g, v^g, \phi^g)$ such that

$$
\begin{align*}
U^g, L_\bar{G}U^g &\in \epsilon^t L^2(Q), \\
B(\bar{U})U^g &= G, \text{ on } \partial Q,
\end{align*}
$$

We then set $U = U^\# + U^g$, and $U^\#$ will be sought as the solution of the homogeneous boundary value problem:

$$
\begin{align*}
L_\bar{G}U^\# &= F - L_\bar{G}U^g, \text{ in } Q, \\
B(\bar{U})U^\# &= 0, \text{ on } \partial Q.
\end{align*}
$$

The existence of $U^\#$ is guaranteed by Proposition 2.3. In conclusion, we have proven the following theorem:

**Theorem 2.4.** Assume that $\tilde{U}$ belongs to $W^{1,\infty}(Q)$ and satisfies the subcritical condition \((3)\). Then there exists $\gamma_0 > 0$ only depending on the $W^{1,\infty}(Q)$-norm of $\tilde{U}$ such that: for all $\gamma > \gamma_0$, for all $F \in \epsilon^t L^2(Q)$ and $G \in \epsilon^t L^2(T \times \mathbb{R})$ satisfying \((19)\), there exists a solution $U \in \epsilon^t L^2(Q)$ satisfying both \((2)\) and \((5)\).

**2.3. An $L^2$ a priori estimate.**

**Lemma 2.5.** Assume that $\tilde{U}$ belongs to $W^{1,\infty}(Q)$ and satisfies the subcritical condition \((3)\). Then we have the a priori estimate

$$
\gamma \|\tilde{U}_\gamma\|_{L^2(Q)} + \epsilon_0 \|\tilde{U}_\gamma\|_{L^2(T \times \mathbb{R})} \leq C_0 \left( \frac{1}{\gamma} \|\bar{F}_\gamma\|_{L^2(Q)} + \|B_0 \bar{U}_\gamma\|_{L^2(T \times \mathbb{R})} + \|B_1 \bar{U}_\gamma\|_{L^2(T \times \mathbb{R})} \right),
$$

for $\bar{U}_\gamma = e^{-\gamma t} U \in H^1(Q)$, $\bar{F}_\gamma = L_\gamma \bar{U}_\gamma$ and all $\gamma > \gamma_0$, where $\gamma_0$ only depends on the bound of the $W^{1,\infty}(Q)$-norm of $\tilde{U}$, and $\epsilon_0, C_0$ only depend on $c_0, c_1, c_2, g$ but are independent of $\tilde{U}$ and $U$.

**Proof.** In the following, we denote by $\langle \cdot, \cdot \rangle$ the $L^2$-scalar product. Multiplying \((8)\) by $S_0$ and taking the scalar product in $L^2(Q)$ with $\tilde{U}_\gamma$ gives

$$
\langle S_0 \partial_t \tilde{U}_\gamma + S_0 F_1 \partial_x \tilde{U}_\gamma + S_0 F_2 \partial_y \tilde{U}_\gamma, \tilde{U}_\gamma \rangle + \langle S_0 \ell(\tilde{U}_\gamma), \tilde{U}_\gamma \rangle + \gamma \langle S_0 \tilde{U}_\gamma, \tilde{U}_\gamma \rangle = \langle S_0 \bar{F}_\gamma, \tilde{U}_\gamma \rangle.
$$

We now calculate

$$
\langle S_0 \ell(\tilde{U}_\gamma), \tilde{U}_\gamma \rangle = 0, \quad \langle S_0 \tilde{U}_\gamma, \tilde{U}_\gamma \rangle \geq \min(1, \frac{g}{c_1}) \|\tilde{U}_\gamma\|_{L^2(Q)}^2,
$$

$$
\langle S_0 \bar{F}_\gamma, \tilde{U}_\gamma \rangle \leq \max(1, \frac{g}{c_0}) \|\bar{F}_\gamma\|_{L^2(Q)} \|\tilde{U}_\gamma\|_{L^2(Q)}.
$$


Integration by parts yields
\begin{align}
\langle S_0 \partial_t \tilde{U}_\gamma, \tilde{U}_\gamma \rangle &= -\frac{1}{2} \langle (S_0)_t \tilde{U}_\gamma, \tilde{U}_\gamma \rangle; \\
\langle S_0 \mathcal{E}_2 \partial_y \tilde{U}_\gamma, \tilde{U}_\gamma \rangle &= -\frac{1}{2} \langle (S_0 \mathcal{E}_2)_y \tilde{U}_\gamma, \tilde{U}_\gamma \rangle,
\end{align}
where the boundary terms disappear at \( y = 0, 1 \) because of the periodic boundary condition in the \( y \)-direction; this also yields
\begin{align}
\langle S_0 \mathcal{E}_1 \partial_x \tilde{U}_\gamma, \tilde{U}_\gamma \rangle &= \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}} (S_0 \mathcal{E}_1 \tilde{U}_\gamma, \tilde{U}_\gamma) \, dt \, dy - \frac{1}{2} \langle (S_0 \mathcal{E}_1)_{x} \tilde{U}_\gamma, \tilde{U}_\gamma \rangle. \tag{25}
\end{align}
We set \( r_1 = \max(1, g/c_0), r_2 = \min(1, g/c_1) \) and
\[ C_1 = \frac{1}{2} \| (S_0(\tilde{U}))_t + (S_0(\tilde{U}) \mathcal{E}_1(\tilde{U}))_x + (S_0(\tilde{U}) \mathcal{E}_2(\tilde{U}))_y \|_{L^\infty(\mathcal{Q})}, \]
and then gathering the calculations (23)-(25), and using relations (7) for estimating the boundary terms in (25) and Young’s inequality for (23)2, we deduce from (22) that for all \( \epsilon > 0 \)
\begin{align}
(r_2 \gamma - C_1) \| \tilde{U}_\gamma \|_{L^2(\mathcal{Q})}^2 + \frac{\epsilon_0}{2} \| \tilde{U}_\gamma \|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \leq 
\frac{r_1^2}{4 \epsilon \gamma} \| \tilde{F}_\gamma \|_{L^2(\mathcal{Q})}^2 + \epsilon \| \tilde{U}_\gamma \|_{L^2(\mathcal{Q})}^2 + \frac{C_0}{2} \| B_0 \tilde{U}_\gamma \|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + \| B_1 \tilde{U}_\gamma \|_{L^2(\mathbb{T} \times \mathbb{R})}^2) \tag{26}
\end{align}
Choosing \( \epsilon = r_2/2, \gamma_0 = 4 C_1/r_2, \) we find that for \( \gamma \geq \gamma_0 \)
\begin{align}
\frac{r_2 \gamma}{4} \| \tilde{U}_\gamma \|_{L^2(\mathcal{Q})}^2 + \frac{\epsilon_0}{2} \| \tilde{U}_\gamma \|_{L^2(\mathbb{T} \times \mathbb{R})}^2 \leq 
\frac{r_1^2}{2 r_2 \gamma} \| \tilde{F}_\gamma \|_{L^2(\mathcal{Q})}^2 + \frac{C_0}{2} \| B_0 \tilde{U}_\gamma \|_{L^2(\mathbb{T} \times \mathbb{R})}^2 + \| B_1 \tilde{U}_\gamma \|_{L^2(\mathbb{T} \times \mathbb{R})}^2) \tag{27}
\end{align}
Renaming the constants \( \epsilon_0 \) and \( C_0 \) yields (21). \( \square \)

**Remark 1.** The estimate (21) is also valid with the whole time interval \( \mathbb{R} \), replaced by the half-line time interval \( (-\infty, T]\). In the half-line time interval case, the only difference is that integration by parts for (24)1 will yield another term at \( t = T \), but the new term is harmless since it is always positive. Therefore, the estimate (26) still holds and hence (21).

### 2.4. Regularity.

**Theorem 2.6.** Suppose that \( \tilde{U} \) belongs to \( \mathcal{C}^\infty(\overline{\mathcal{Q}}) \) and is constant outside a compact subset \( \mathcal{O} \) of \( \overline{\mathcal{Q}} \), and satisfies the subcritical condition (3). Then for all integers \( m \geq -1 \), the four conditions
\[ V \in H^m(\mathcal{Q}) \cap L^2(\mathcal{Q}), \quad L_{\delta t} V \in H^{m+1}(\mathcal{Q}), \quad \forall \gamma > \gamma_0, \]
\[ V|_{x=0,1} \in H^m(\mathbb{T} \times \mathbb{R}), \quad B_0(\tilde{U}) V|_{x=0}, B_1(\tilde{U}) V|_{x=1} \in H^{m+1}(\mathbb{T} \times \mathbb{R}), \]
imply
\[ V \in H^{m+1}(\mathcal{Q}) \text{ and } V|_{x=0,1} \in H^{m+1}(\mathbb{T} \times \mathbb{R}). \]

**Proof.** A first useful remark is that it suffices to show that \( V \) belongs to the space \( L^2_2(0,1;H^{m+1}(\mathbb{T} \times \mathbb{R})) \). Indeed, we have
\begin{align}
V_x = \mathcal{E}_1(\tilde{U})^{-1} (L_{\gamma,\tilde{U}} V - \partial_t V - \mathcal{E}_2(\tilde{U}) \partial_y V - \ell(V) - \gamma V) \in L^2_2(0,1;H^{m+1}(\mathbb{T} \times \mathbb{R})), \tag{28}
\end{align}
and then we show by induction that $\partial^k_x V \in L^2_x(0, 1; H^{m-k+1}(T \times \mathbb{R}))$ for $k = 2, \cdots, m + 1$, which implies that $V \in H^{m+1}(Q)$.

Let $\sigma_\epsilon(y, t)$ be the mollifier defined in Subsection A.4 with $n_1 = n_2 = 1$, $y = x_1$ and $t = x_2$, and set $V_\epsilon = \sigma_\epsilon \ast V$. By construction, the function $V_\epsilon$ belongs to $L^2_x(0, 1; H^{+\infty}(T \times \mathbb{R}))$, and by (28):

$$V_{\epsilon,x} = \sigma_\epsilon \ast [E_1(\tilde{U})^{-1}(L_{\gamma,\tilde{U}} V - \ell(V) - \gamma V)] + E_1(\tilde{U})^{-1} \partial_t V_\epsilon$$

$$+ E_1(\tilde{U})^{-1} E_2(\tilde{U}) \partial_y V_\epsilon + [E_1(\tilde{U})^{-1} \partial_t, \sigma_\epsilon] V + [E_1(\tilde{U})^{-1} E_2(\tilde{U}) \partial_y, \sigma_\epsilon] V$$

(29)

belongs to $L^2(Q)$: for the first term we use the fact that the function in the bracket is at least $L^2$ and that $\sigma_\epsilon$ is a bounded operator on $L^2$; then the second and third terms belong to $L^2$ since $V_\epsilon \in L^2_x(0, 1; H^{+\infty}(T \times \mathbb{R}))$; for the last two terms, we apply Friedrichs’ Lemma ([6], or Lemma B.1 iv) in the $(y, t)$ variables. Hence, we conclude that $V_{\epsilon,x} \in L^2(Q)$, which implies that $V_\epsilon$ actually belongs to $H^1(Q)$. Therefore, we can apply the $L^2$-estimate (21) to $V_\epsilon$, which reads

$$\gamma \|V_\epsilon\|^2_{L^2(Q)} + \epsilon_0 \|V_\epsilon|_{x=0,1}\|^2_{L^2(T \times \mathbb{R})}$$

$$\leq C_0 \left( \frac{1}{\gamma} \|L_{\gamma,\tilde{U}} V_\epsilon\|^2_{L^2(T \times \mathbb{R})} + \|B_0 V_\epsilon|_{x=0}\|^2_{L^2(T \times \mathbb{R})} + \|B_1 V_\epsilon|_{x=1}\|^2_{L^2(T \times \mathbb{R})} \right).$$

(30)

Multiplying by $\epsilon^{-2m-3}(1+\theta^2/\epsilon^2)^{-1}$, integrating in $\epsilon \in (0, 1]$, and using the notation $P_{m, \theta}$ as in Proposition A.7, we arrive at

$$\gamma \int_0^1 P_{m, \theta}(V) dx + \epsilon_0 P_{m, \theta}(V|_{x=0,1})$$

$$\leq C_0 \left( \frac{1}{\gamma} \int_0^1 P_{m, \theta}(L_{\gamma,\tilde{U}} V) dx + P_{m, \theta}(B_0 V|_{x=0}) + P_{m, \theta}(B_1 V|_{x=1})$$

$$+ \frac{1}{\gamma} \int_0^1 \int_0^1 \|E_1(\tilde{U}) \partial_x, \sigma_\epsilon \ast V\|_{L^2(T \times \mathbb{R})}$$

$$+ \|E_2(\tilde{U}) \partial_y, \sigma_\epsilon \ast V\|_{L^2(T \times \mathbb{R})} \epsilon^{-2m-2}(1+\theta^2/\epsilon^2)^{-1} \frac{d\epsilon}{\epsilon} dx$$

$$\leq C_0 \left( \int_0^1 \|B_0, \sigma_\epsilon \ast V\|_{x=0}\|^2_{L^2(T \times \mathbb{R})} + \|B_1, \sigma_\epsilon \ast V\|_{x=1}\|^2_{L^2(T \times \mathbb{R})} \epsilon^{-2m-2}(1+\theta^2/\epsilon^2)^{-1} \frac{d\epsilon}{\epsilon} \right).$$

(31)

where we have used the relations $B_i V_\epsilon = \sigma_\epsilon \ast B_i V + [B_i, \sigma_\epsilon \ast V]$ for $i = 0, 1$, and

$L_{\gamma,\tilde{U}} V_\epsilon = \sigma_\epsilon \ast L_{\gamma,\tilde{U}} V + [E_1(\tilde{U}) \partial_x, \sigma_\epsilon \ast V] + [E_2(\tilde{U}) \partial_y, \sigma_\epsilon \ast V].$

Viewing $x$ as a parameter varying on the closed interval $[0, 1]$, applying Proposition A.8 with $x_1 = y, x_2 = t, n_1 = n_2 = 1, s = m, |\alpha| = 1$ or 0 to the four commutators gives

$$\gamma \int_0^1 P_{m, \theta}(V) dx + \epsilon_0 P_{m, \theta}(V|_{x=0,1})$$

$$\leq C_0 \left( \frac{1}{\gamma} \int_0^1 P_{m, \theta}(L_{\gamma,\tilde{U}} V) dx + P_{m, \theta}(B_0 V|_{x=0}) + P_{m, \theta}(B_1 V|_{x=1})$$

$$+ C \int_0^1 \|V\|^2_{m, \theta} dx + \|V|_{x=0,1}\|^2_{m-1, \theta} \right).$$

(31)
Theorem 2.6. Let \( m \geq 3 \), \( \beta > 0 \) and \( \kappa_0 > 0 \) be constants such that \( \kappa_0 \) is independent of \( \gamma \) and \( \beta \). Let \( \hat{U} \) be a priori estimate, satisfy the subcritical condition (3) and \( \hat{U} \in \mathcal{H}_\gamma^m(\mathbb{Q}) \). Then there exists \( \gamma_m = \gamma_m(M) \geq \gamma_0 > 0 \) such that, for all \( \gamma > \gamma_m \), for all \( U \in \mathcal{H}_\gamma^m(\mathbb{Q}) \), we have the a priori estimate

\[
\gamma \| U \|_{\mathcal{H}_\gamma^m(\mathbb{Q})}^2 + \epsilon_0 \| U \|_{\mathbb{Q}}^2 \leq C(M) \left( \frac{1}{\gamma} \| F \|_{\mathcal{H}_\gamma^m(\mathbb{Q})}^2 + \sum_{i=0}^1 \| B_i U \|_{\mathbb{Q}}^2 \right),
\]

where \( \epsilon_0 \) is that of Lemma 2.5, and \( \gamma_m(M) \), \( C(M) \) may also depend on the constants \( c_0, c_1, c_2, g, \kappa_0 \) and the set \( \mathcal{O} \), but are independent of \( U \).
Proof. Tangential derivatives: For simplicity, we denote by $L^2(H^k)$ (resp. $L^2(H^K)$) the space $L^2(0,1; H^k_y;\mathbb{T} \times \mathbb{R}$) (resp. $L^2(0,1; H^k_y;\mathbb{T} \times \mathbb{R}$)), and denote by $C(M)$ some generic constant depending only on $M$, which may also depend on $c_0, c_1, c_2, g, \kappa_0,$ and $O$, but is independent of $U$, the values of which may vary from one place to another.

Applying $\partial^\beta = \partial^\beta_1 \partial^\beta_2$ with $0 \leq |\beta| = \beta_1 + \beta_2 \leq m$ to (2) yields

$$\partial_t \partial^\beta U + E_1 \partial_x \partial^\beta U + E_2 \partial_y \partial^\beta U + [\partial^\beta, E_1] \partial_x U + [\partial^\beta, E_2] \partial_y U + \ell \partial^\beta U = \partial^\beta F; \quad (37)$$

then multiplying by $e^{-\gamma t}$ on both sides of (37) and using the notation $L_{\gamma, U}$, we find that

$$L_{\gamma, U}(e^{-\gamma t} \partial^\beta U) = e^{-\gamma t} \partial^\beta F - e^{-\gamma t}[\partial^\beta, E_1] \partial_x U - e^{-\gamma t}[\partial^\beta, E_2] \partial_y U$$

$$=: \tilde{F}. \quad (38)$$

Using the $L^2$-estimate (21) for (38), we obtain

$$\gamma \|e^{-\gamma t} \partial^\beta U\|_{L^2(Q)}^2 + \epsilon_0 \|e^{-\gamma t} \partial^\beta U\|_{L^2(T \times \mathbb{R})} \leq C_0 \left( \frac{1}{\gamma} \|\tilde{F}\|_{L^2(Q)}^2 \right)$$

$$+ \|B_0(e^{-\gamma t} \partial^\beta U)\|_{L^2(T \times \mathbb{R})}^2 + \|B_1(e^{-\gamma t} \partial^\beta U)\|_{L^2(T \times \mathbb{R})}^2 \quad (39)$$

We now need to estimate the source term $\tilde{F}$ and the boundary terms. We first focus on the source term $\tilde{F}$. A slight difficulty arises here since the matrices $E_1(U), E_2(\tilde{U})$ do not belong to $H^m$; in order to overcome this difficulty, we split these operators into two parts as follows:

$$E_1(\tilde{U}) = E_1^c + E_1^w(\tilde{U} - \kappa_0), \quad E_2(\tilde{U}) = E_2^c + E_2^w(\tilde{U} - \kappa_0), \quad (40)$$

where

$$E_1^c = \begin{pmatrix} \kappa_0 & 0 & g \\ 0 & \kappa_0 & 0 \\ \kappa_0 & 0 & \kappa_0 \end{pmatrix}, \quad E_1^w(\tilde{U} - \kappa_0) = \begin{pmatrix} \tilde{u} - \kappa_0 & 0 & 0 \\ 0 & \tilde{u} - \kappa_0 & 0 \\ \tilde{\phi} - \kappa_0 & 0 & \tilde{u} - \kappa_0 \end{pmatrix},$$

$$E_2^c = \begin{pmatrix} \kappa_0 & 0 & 0 \\ 0 & \kappa_0 & g \\ 0 & \kappa_0 & \kappa_0 \end{pmatrix}, \quad E_2^w(\tilde{U} - \kappa_0) = \begin{pmatrix} \tilde{v} - \kappa_0 & 0 & 0 \\ 0 & \tilde{v} - \kappa_0 & 0 \\ \tilde{\phi} - \kappa_0 & 0 & \tilde{v} - \kappa_0 \end{pmatrix},$$

where $E_i^c$ (resp. $E_i^w$) captures the behavior of $E_i$ outside (resp. inside) of $O$ for $i = 1,2$.

Now since the $E_i^c$, $(i = 1, 2)$ are constant matrices, they commute with $\partial^\beta$. Hence, we find $[\partial^\beta, E_i(\tilde{U})] = [\partial^\beta, E_i^c(\tilde{U} - \kappa_0)]$. Using Lemma B.1 ii) for $E_i^c(\tilde{U} - \kappa_0)$, we obtain that

$$\|E_i^c(\tilde{U} - \kappa_0)\|_{H^m(Q)} \leq C \|\tilde{U} - \kappa_0\|_{H^m(Q)} \leq C(M), \quad \text{for } i = 1,2; \quad (41)$$

then applying Lemma B.2 ii) on the second commutator in $\tilde{F}$ only in the $(y,t)$ variables yields

$$\|e^{-\gamma t}[\partial^\beta, E_2(\tilde{U})] \partial_y U\|_{L^2(Q)} = \|e^{-\gamma t}[\partial^\beta, E_2^c(\tilde{U} - \kappa_0)] \partial_y U\|_{L^2(Q)}$$

$$\leq C \|E_2^c(\tilde{U} - \kappa_0)\|_{H^m(Q)} \|\partial_y U\|_{L^2(Q)} \quad (42)$$

$$\leq C(M) \|U\|_{L^2(H^k_y)}. \quad (43)$$
For the first commutator in \( \tilde{F} \), we similarly have
\[
\|e^{-\gamma t}[\partial^3, \mathcal{E}_1(\tilde{U})]\partial_x U\|_{L^2(Q)} \leq C(M)\|\partial_x U\|_{L^2(H^{3,1})}. \tag{43}
\]
Note that (43) does not have the exact form as (42) since the \( x \)-direction is not the tangential direction. In order to further estimate the term \( \|\partial_x U\|_{L^2(H^{3,1})} \), we first deduce from (2) that
\[
\partial_x U = \mathcal{E}_1(\tilde{U})^{-1}(F - \ell(U) - \partial_t U - \mathcal{E}_2(\tilde{U})\partial_y U). \tag{44}
\]
Direct computation shows that
\[
\mathcal{E}_1(\tilde{U})^{-1} = \begin{pmatrix}
\frac{\hat{u}}{\hat{u}^2 - g\phi} & 0 & \frac{g}{\hat{u}^2 - g\phi} \\
0 & \frac{1}{\hat{u}} & 0 \\
\frac{\hat{u}}{\hat{u}^2 - g\phi} & 0 & \frac{\hat{u}}{\hat{u}^2 - g\phi}
\end{pmatrix}.
\]
We claim that \(1/\hat{u} - 1/\kappa_0\) belongs to \(H^m(Q)\). By assumption, we have that \(\hat{u}\) belongs to \(H^m(O)\) and is bounded away from 0, and the domain \(O\) is compact, and thus bounded. Then applying Lemma B.1 ii) to \(\hat{u}\) shows that \(1/\hat{u} \in H^m(O)\), which implies that \(1/\hat{u} - 1/\kappa_0 \in H^m(Q)\) since \(1/\hat{u} - 1/\kappa_0\) is zero outside of \(O\). Using similar arguments to each entry of the matrix \(\mathcal{E}_1(\tilde{U})^{-1} - (\mathcal{E}_1^\gamma)^{-1}\), we obtain that
\[
\|\mathcal{E}_1(\tilde{U})^{-1} - (\mathcal{E}_1^\gamma)^{-1}\|_{H^m(Q)} \leq C(M). \tag{45}
\]
We then rewrite (44) as:
\[
\partial_x U = ((\mathcal{E}_1^\gamma)^{-1} + \mathcal{E}_1(\tilde{U})^{-1} - (\mathcal{E}_1^\gamma)^{-1}) \left(F - \ell(U) - \partial_t U - (\mathcal{E}_2 + \mathcal{E}_2(\tilde{U} - \kappa_0))\partial_y U\right), \tag{46}
\]
and then applying Lemma B.2 ii) in the \((y, t)\) variables and using (41) and (45), we find that
\[
\|\partial_x U\|_{L^2(H^{3,1})} \leq \left(\|\mathcal{E}_1^\gamma\|^{-1} + \|\mathcal{E}_1(\tilde{U})^{-1} - (\mathcal{E}_1^\gamma)^{-1}\|_{L^2(H^{m})}\right)\left(\|F\|_{L^2(H^{3,1})} + \|\mathcal{E}_2\|_{L^2(H^{3,1})} + \|\partial_y U\|_{L^2(H^{3,1})}\right) \tag{47}
\]
\[
\leq C(M)\left(\|F\|_{L^2(H^{3,1})} + \|U\|_{L^2(H^{3,1})} + \|\partial_y U\|_{L^2(H^{3,1})}\right).
\]
Combining the estimates (42)-(43) and (47), we obtain the estimate for \(\tilde{F}\):
\[
\|\tilde{F}\|^2_{L^2(Q)} \leq \|e^{-\gamma t}\partial^3 F\|^2_{L^2(Q)} + C(M)\left(\|F\|_{L^2(H^{3,1})} + \|U\|_{L^2(H^{3,1})}\right) \tag{48}
\]
We now turn to the boundary terms. Noting that the boundary operators
\[
B_0(\tilde{U}), B_1(\tilde{U}) \text{ are not } H^m, \text{ like } \mathcal{E}_1(\tilde{U}), \mathcal{E}_2(\tilde{U}), \text{ we also split them into two parts as follows:}
\]
\[
B_0(\tilde{U}) = B_0^c + B_0^\gamma (\tilde{U} - \kappa_0), \quad B_1(\tilde{U}) = B_1^c + B_1^\gamma (\tilde{U} - \kappa_0),
\]
where
\[ B_0^c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_0^c(\bar{U} - \kappa_0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]
\[ B_1^c = (1, 0, -\sqrt{g/\kappa_0}), \quad B_1^c(\bar{U} - \kappa_0) = (0, 0, -\sqrt{g/\phi + g/\kappa_0}). \]

Arguing similarly as for (45), we find that for \( i = 0, 1 \):
\[
\|B_i^c(\bar{U} - \kappa_0)\|_{H^m(T \times R)} \leq \|\sqrt{g/\phi - \sqrt{g/\kappa_0}}\|_{H^m(T \times R)} \leq C(M). \quad (49)
\]

Now, since the \( B_i^c, (i = 0, 1) \) are constant matrices, they commute with \( \partial^\beta \). Hence, the boundary terms in (39) can be rewritten as
\[
B_i(\bar{U})(e^{-\gamma t} \partial^\beta U) = e^{-\gamma t} \partial^\beta (B_i(\bar{U})U) - e^{-\gamma t} \partial^\beta (B_i(\bar{U}))U
= e^{-\gamma t} \partial^\beta (B_i(\bar{U})U) - e^{-\gamma t} \partial^\beta (B_i(\bar{U} - \kappa_0))U. \quad (50)
\]

For \( i = 0, 1 \), using the commutator estimates in Lemma B.2 ii), we obtain that
\[
\|e^{-\gamma t} \partial^\beta, B_i^c(\bar{U} - \kappa_0)\|_{L^2(T \times R)} \leq \|e^{-\gamma t} \partial^\beta (B_i(\bar{U})U)\|_{L^2(T \times R)} + 1 \gamma C(M) \|U\|_{H^{m,b}} \leq \|e^{-\gamma t} \partial^\beta (\bar{U})U\|_{L^2(T \times R)} + \frac{1}{\gamma^2} C(M) \|U\|_{H^{m,b}} \quad (51)
\]

(by the definition of the \( H^{m,b} \) norm, we multiply (39) by \( \gamma^{2(m-|\beta|)} \), and sum on \( \beta \); noticing that
\[
\sum_{|\beta| \leq m} \gamma^{2(m-|\beta|)} \|U\|_{L^2(H^{\beta})}^2 \leq C \|U\|_{L^2(H^{\gamma})}^2, \quad (53)
\]
for some constant \( C \) independent of \( U \) and \( \gamma \), and using the estimates (48) and (52) we then arrive at
\[
\gamma \|U\|_{L^2(H^{\gamma})}^2 + \epsilon_0 \|U\|_{x=0,1} H^{\gamma}(T \times R) \leq C_0 \left( \frac{1}{\gamma} \|F\|_{L^2(H^{\gamma})}^2 + \frac{1}{\gamma^2} \|\bar{U}\|_{L^2(H^{\gamma})}^2 \right) + \|U\|_{L^2(H^{\gamma})}^2 \]
\[
+ \sum_{i=0} \|B_i U\|_{x=0,1} H^{\gamma}(T \times R) + \frac{1}{\gamma^2} C(\gamma) \|U\|_{x=0,1} H^{\gamma}(T \times R), \quad (54)
\]
which implies that
\[
\gamma \|U\|_{L^2(H^{\gamma})}^2 + \epsilon_0 \|U\|_{x=0,1} H^{\gamma}(T \times R) \leq C(M) \left( \frac{1}{\gamma} \|F\|_{L^2(H^{\gamma})}^2 \right) + \sum_{i=0} \|B_i U\|_{x=0,1} H^{\gamma}(T \times R), \quad (54)
\]
for \( \gamma \geq \sqrt{2C_0 C(M)/\epsilon_0} \).
Normal derivatives: Choosing $|\beta| = m$ in (47), we readily have
\[
\| \partial_x U \|^2_{L^2(H^{m-1}_\gamma)} \leq C(M) \left( \| F \|^2_{L^2(H^{m-1}_\gamma)} + \| U \|^2_{L^2(H^m_\gamma)} \right) 
\leq C(M) \left( \frac{1}{\gamma^2} \| F \|^2_{L^2(H^m_\gamma)} + \| U \|^2_{L^2(H^m_\gamma)} \right).
\]

Multiplying (55) by $\gamma$ and using estimate (54) for the last term implies that
\[
\gamma \| \partial_x U \|^2_{L^2(H^{m-1}_\gamma)} \leq C(M) \left( \frac{1}{\gamma^2} \| F \|^2_{L^2(H^m_\gamma)} + \sum_{i=0}^1 \| B_i U \|_{x=i}^2 \right)
\]
(56)

We also need to estimate the terms involving the derivatives $\partial_x^j$ for $j = 1, \ldots, m$, and we will show it by a recurrent argument. For that purpose, we assume that we have the following energy estimate for $\partial_x^j U$:
\[
\gamma \| \partial_x^j U \|^2_{L^2(H^{m-j}_\gamma)} \leq C(M) \left( \frac{1}{\gamma^2} \| F \|^2_{H^m_\gamma} + \frac{1}{\gamma} \| U \|^2_{H^m_\gamma} + \sum_{i=0}^1 \| B_i U \|_{x=i}^2 \right). 
\]
(57)

Note that (57) is true for $j = 1$. We then apply $\partial_x^j$ to (44), and obtain that
\[
\partial_x^{j+1} U = \partial_x^j (E_1(\hat{U})^{-1}(F - \ell(U) - \partial_t U - E_2(\hat{U})\partial_y U))
= \partial_x^j (E_1(\hat{U})^{-1} F) - \partial_x^j (E_1(\hat{U})^{-1} \ell(U)) - E_1(\hat{U})^{-1} \partial_t \partial_x^j U
- E_1(\hat{U})^{-1} E_2(\hat{U}) \partial_y \partial_x^j U - [\partial_x^j, E_1(\hat{U})^{-1}] \partial_t U - [\partial_x^j, E_1(\hat{U})^{-1} E_2(\hat{U})] \partial_y U.
\]
(58)

We then need to estimate the six terms in the right-hand side of (58). In the following estimates, we split $E_1, E_2$ into two parts as defined by (40). For the first term, we find:
\[
\| \partial_x^j (E_1^{-1} F) \|^2_{L^2(H^{m-j}_\gamma)} \leq \| (E_1^{-1} + E_1(\hat{U})^{-1} - (E_1^{-1}) F \|^2_{H^{m-j}_\gamma} 
\leq \text{(using Lemma B.2 i))}
\leq C \left( \| (E_1^{-1}) \|^2 + \| E_1(\hat{U})^{-1} - (E_1^{-1}) \|^2_{H^m_\gamma} \right) \| F \|^2_{H^{m-j}_\gamma} 
\leq \frac{C(M)}{\gamma^2} \| F \|^2_{H^m_\gamma}.
\]

The second term can be estimated similarly as the first term, and we have
\[
\| \partial_x^j (E_1^{-1} \ell(U)) \|^2_{L^2(H^{m-j}_\gamma)} \leq \frac{C(M)}{\gamma^2} \| U \|^2_{H^m_\gamma}.
\]
(59)

Using Lemma B.2 i) only in the $(y, t)$ variables, we then estimate the third term as follows:
\[
\| E_1(\hat{U})^{-1} \partial_t \partial_x^j U \|^2_{L^2(H^{m-j}_\gamma)} 
\leq C \left( \| (E_1^{-1}) \|^2 + \| E_1(\hat{U})^{-1} - (E_1^{-1}) \|^2_{L^2(H^m_\gamma)} \right) \| \partial_t \partial_x^j U \|^2_{L^2(H^{m-j}_\gamma)}
\leq C(M) \| \partial_x^j U \|^2_{L^2(H^{m-j}_\gamma)}.
\]
(60)

We estimate the fourth term similarly as the third term but here we need to split both $E_1(\hat{U})^{-1}$ and $E_2(\hat{U})$, and we split $E_2$ into two parts as defined in (40) and split $(E_1^{-1})$ into the constant matrix $(E_1^{-1})$ and the matrix $(E_1^{-1} - (E_1^{-1}))$, we then find
\[
\| E_1(\hat{U})^{-1} E_2(\hat{U}) \partial_y \partial_x^j U \|^2_{L^2(H^{m-j}_\gamma)} \leq C(M) \| \partial_x^j U \|^2_{L^2(H^{m-j}_\gamma)}.
\]
(61)
For the fifth term, we find
\[ \| \partial_x^j \mathcal{E}_1(\tilde{U})^{-1} \partial_t U \|_{L^2(\mathcal{H}_m^{m-j-1})} \leq \| \partial_x^j \mathcal{E}_1(\tilde{U})^{-1} - (\mathcal{E}_1^{-1} - \mathcal{E}_1')^{-1} \partial_t U \|_{L^2(\mathcal{H}_m^{m-j-1})} \]
\[ \leq (\text{using Lemma B.2 iii)}) \]
\[ \leq C \| \mathcal{E}_1(\tilde{U})^{-1} - (\mathcal{E}_1^{-1} - \mathcal{E}_1')^{-1} \|_{H^m(Q)} \| \partial_t U \|_{L^2(\mathcal{H}_m^{m-j-1})} \]
\[ \leq C(M) \| \partial_x^j \mathcal{E}_1(\tilde{U})^{-1} \partial_t U \|_{L^2(\mathcal{H}_m^{m-j-1})}. \]

Splitting both \( \mathcal{E}_1(\tilde{U})^{-1} \) and \( \mathcal{E}_2(\tilde{U}) \) as before and using Lemma B.2 iii), we have a similar estimate for the last (sixth) term:
\[ \| \partial_x^j \mathcal{E}_1(\tilde{U})^{-1} \mathcal{E}_2(\tilde{U}) \partial_t U \|_{L^2(\mathcal{H}_m^{m-j-1})} \leq \frac{C(M)}{\gamma^2} \| U \|_{L^2(\mathcal{H}_m^{m-j-1})}. \]  

Gathering all the estimates (59)-(64), we find that
\[ \| \partial_x^{j+1} U \|_{L^2(\mathcal{H}_m^{m-j-1})} \leq \frac{C(M)}{\gamma^2} \left( \| F \|_{L^2(\mathcal{H}_m^{m}(Q))} + \| U \|_{L^2(\mathcal{H}_m^{m}(Q))} \right) + C(M) \| \partial_x^j U \|_{L^2(\mathcal{H}_m^{m-j-1})}, \]
which, together with assumption (57), implies that:
\[ \gamma \| \partial_x^{j+1} U \|_{L^2(\mathcal{H}_m^{m-j-1})} \leq C(M) \left( \frac{1}{\gamma} \| F \|_{L^2(\mathcal{H}_m^{m}(Q))} + \frac{1}{\gamma} \| U \|_{L^2(\mathcal{H}_m^{m}(Q))} + \sum_{i=0}^{j+1} \| B_i U \|_{L^2(\mathcal{H}_m^{m}(Q))} \right), \]
which is what we wanted to prove by induction.

Now, summing over \( j = 1, \ldots, m \) for (57), and using (54) for \( j = 0 \), we obtain the following estimate:
\[ \gamma \| U \|_{L^2(\mathcal{H}_m^{m}(Q))} + \epsilon_0 \| U \|_{L^2(\mathcal{H}_m^{m}(Q))} \leq C(M) \left( \frac{1}{\gamma} \| F \|_{L^2(\mathcal{H}_m^{m}(Q))} + \frac{1}{\gamma} \| U \|_{L^2(\mathcal{H}_m^{m}(Q))} \right) \]
\[ + \sum_{i=0}^{j+1} \| B_i U \|_{L^2(\mathcal{H}_m^{m}(Q))}. \]

Choosing \( \gamma \) large enough to absorb the second term in (65) gives the estimate (36) announced. We thus completed the proof of Lemma 2.7.

Remark 2. The estimate (36) is also valid with the whole time interval \( \mathbb{R}_t \) replaced by the half-line time interval \( (-\infty, T]_t \) (see also Remark 1). 

2.6. \( H^m \) well-posedness. Our main theorems in this section are the following.

Theorem 2.8. Assume that \( \tilde{U} \) belongs to \( C^\infty(\mathcal{Q}) \) and is constant outside a compact subset \( \mathcal{O} \) of \( \mathcal{Q} \), and satisfies the subcritical condition (3). Then there exists \( \gamma_0 \geq 1 \) only depending on \( \tilde{U} \), such that for all \( \gamma > \gamma_0 \), for all \( F \in e^{\gamma t}L^2(Q) \) and all \( G = (g_1, g_2, g_3)^t \in e^{\gamma t}L^2(T \times \mathbb{R}) \), there exists a unique solution \( \tilde{U} \in e^{\gamma t}L^2(Q) \) which satisfies (2) and (3) and \( \tilde{U}_\gamma = e^{-\gamma t}U \) enjoys the estimate (21), i.e.
\[ \gamma \| \tilde{U}_\gamma \|_{L^2(\mathcal{H}_m^{m}(Q))} + \epsilon_0 \| \tilde{U}_\gamma \|_{L^2(\mathcal{H}_m^{m}(Q))} \leq C_0 \left( \frac{1}{\gamma} \| F \|_{L^2(Q)} + \| B_0 \tilde{U}_\gamma \|_{L^2(T \times \mathbb{R})} \right) \]
\[ + \| B_1 \tilde{U}_\gamma \|_{L^2(T \times \mathbb{R})}. \]

Furthermore, for any \( m \in \mathbb{N} \), there exists \( \gamma_m > \gamma_0 \) depending only on \( \tilde{U} \) such that, for all \( \gamma > \gamma_m \), for all \( F \in e^{\gamma t}H^m(Q) \), \( G = (g_1, g_2, g_3)^t \in e^{\gamma t}H^m(T \times \mathbb{R}) \), the solution \( U \) of (2) and (3) belongs to \( e^{\gamma t}H^m(Q) \) and its trace belongs to \( e^{\gamma t}H^m(T \times \mathbb{R}) \).
Proof. **Existence** By Theorem 2.4, there exists $U \in e^{\gamma t}L^2(Q)$ satisfying (2) and (5) if $F \in e^{\gamma t}L^2(Q)$ and $G \in e^{\gamma t}L^2(T \times \mathbb{R})$. Noticing that $U|_{x=0,1}$ belongs to $e^{\gamma t}H^{-1}(T \times \mathbb{R})$ thanks to Proposition 2.1, Theorem 2.6 applied with $m=-1$ and $V = \tilde{U}_e$ readily implies that $U|_{x=0,1}$ belongs to $e^{\gamma t}L^2(T \times \mathbb{R})$. Now, if additionally $F \in e^{\gamma t}H^1(Q)$ and $G \in e^{\gamma t}H^1(T \times \mathbb{R})$, we can apply Theorem 2.6 again with $m=0$ and $V = \tilde{U}_e$, and thus obtain that $U$ belongs to $e^{\gamma t}H^1(Q)$ and its traces to $e^{\gamma t}H^1(T \times \mathbb{R})$. By induction, we thus show that $U$ and its traces inherit exactly the Sobolev order of $F$ and $G$.

**Uniqueness** By linearity it suffices to prove that the only $e^{\gamma t}L^2$ solution of the homogeneous problem

$$
\begin{align*}
\begin{cases}
L_0 U = 0, & \text{in } Q \\
B(\tilde{U})U = 0, & \text{on } \partial Q,
\end{cases}
\end{align*}
$$

is 0. If $U \in e^{\gamma t}L^2(Q)$ solves (67), then $U$ belongs to $e^{\gamma t}H^1(Q)$ (because the vanishing right-hand sides belong to $H^1$) thanks to Theorem 2.6, and therefore $U$ satisfies the energy estimate (66) with zero sourcing term and zero boundary terms, which of course implies $U = 0$ almost everywhere.

**Energy estimate** To complete the proof of Theorem 2.8 it remains to show that the estimate (66) is valid as soon as $U$ belongs to $e^{\gamma t}L^2(Q)$. To prove this fact, we regularize the data $F$ and $G$ and use the regularity part of Theorem 2.8 together with the uniqueness of the solution. For any $F \in e^{\gamma t}L^2(Q)$ and $G \in e^{\gamma t}L^2(T \times \mathbb{R})$, there exist $F_\epsilon \in C^\infty(Q)$ and $G_\epsilon \in C^\infty(T \times \mathbb{R})$ such that

$$
e^{-\gamma t}F_\epsilon \to e^{-\gamma t}F, \text{ in } L^2(Q) ; \quad e^{-\gamma t}G_\epsilon \to e^{-\gamma t}G, \text{ in } L^2(T \times \mathbb{R}).$$

For each $\epsilon > 0$, there exists a unique $U_\epsilon \in e^{\gamma t}H^1(Q)$ with $\gamma \geq \gamma_1$ solving the regularized problem

$$
L_0 U_\epsilon = F_\epsilon, \text{ in } Q, \quad B(\tilde{U})U_\epsilon = G_\epsilon, \text{ on } \partial Q.
$$

The energy estimate (66) is valid for the (smooth) difference $(U_\epsilon - U_\epsilon^\prime)$ and thus shows that both $(U_\epsilon)_{\epsilon > 0}$ and $(U_{\epsilon}|_{x=0,1})_{\epsilon > 0}$ are Cauchy sequences in $e^{\gamma t}L^2(Q)$ and $e^{\gamma t}L^2(T \times \mathbb{R})$, respectively. Therefore, there exist $U \in e^{\gamma t}L^2(Q)$ and $U_0 \in e^{\gamma t}L^2(T \times \mathbb{R})$ such that

$$
\|e^{-\gamma t}(U_\epsilon - U)\|_{L^2(Q)} \to 0, \text{ and } \|e^{-\gamma t}(U_{\epsilon}|_{x=0,1} - U_0)\|_{L^2(T \times \mathbb{R})} \to 0, \text{ as } \epsilon \to 0.
$$

By construction, $L_0 U_\epsilon = F_\epsilon$ converges to $L_0 U = F$ (at least) in the sense of distributions. Consequently, by Proposition 2.1 the trace $U_{\epsilon}|_{x=0,1}$ converges to $U|_{x=0,1}$ in $e^{\gamma t}H^{-1}(T \times \mathbb{R})$, hence by uniqueness of limits in the sense of distributions, $U_{\epsilon}|_{x=0,1} \to U|_{x=0,1}$. In the limit, we thus find $B(\tilde{U})U = G$ on $\partial Q$. Therefore, $U$ satisfies both (2) and (5) as $U$, and then $U = \tilde{U}$ by uniqueness of the solution. Finally, passing to the limit (taking lim inf) in the energy estimate (66) for such $U_\epsilon$, we obtain (66) for $U$.

**Remark 3.** The previous regularization method and Lemma 2.7 applied to $U_\epsilon$, show that there exists $C(M) > 0$ so that for $F \in H^m_\Omega(Q)$ and $G \in H^m_\Omega(T \times \mathbb{R})$ with $\gamma > \gamma_m$, the solution $U$ of the boundary value problem (2),(5) satisfies

$$
\gamma\|U\|_{H^m_\Omega(Q)} + \epsilon_0\|U|_{x=0,1}\|_{H^m_\Omega(T \times \mathbb{R})}^2 \leq C(M)\left(\frac{1}{\gamma}\|F\|_{H^m_\Omega(Q)}^2 + \|G\|_{H^m_\Omega(T \times \mathbb{R})}^2\right).
$$

The estimate (68) is also valid with the whole time interval $\mathbb{R}_t$ replaced by the half-line time interval $(-\infty, T]_t$. 

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Theorem 2.9. Let \( m \geq 3 \) be an integer and \( \kappa_0 \> 0 \) be a fixed read number. Assume that \( \hat{U} - \kappa_0 \) has compact support \( Q \) in \( \mathbb{R} \), and that \( \hat{U} \) satisfies the subcritical condition (3), and that \( \hat{U} - \kappa_0 \) belongs to \( H^m(Q) \) and its traces \( (\hat{U} - \kappa_0)|_{x=0,1} \) belong to \( H^m(T \times \mathbb{R}) \) and satisfy:

\[
\| \hat{U} - \kappa_0 \|_{H^m(Q)} \leq M, \quad \| (\hat{U} - \kappa_0)|_{x=0,1} \|_{H^m(T \times \mathbb{R})} \leq M.
\]

Then there exists \( \gamma_m > 0 \) only depending on \( M \) such that: for all \( \gamma > \gamma_m \), for all \( F \in H^m(Q) \) and \( G \in H^m(T \times \mathbb{R}) \), there exists a unique function \( U \in H^m(Q) \) which satisfies both (2) and (5) and the energy estimates (68) and (66).

Proof. By regularization, there exists \( \hat{U}_\epsilon \in C^\infty(Q) \) such that

\[
\hat{U}_\epsilon - \kappa_0 \rightarrow \hat{U} - \kappa_0, \text{ in } H^m(Q), \quad (\hat{U}_\epsilon - \kappa_0)|_{x=0,1} \rightarrow (\hat{U} - \kappa_0)|_{x=0,1}, \text{ in } H^m(T \times \mathbb{R}).
\]

For \( \epsilon > 0 \) small enough, we may assume that

\[
\| \hat{U}_\epsilon - \kappa_0 \|_{H^m(Q)} \leq 2M, \quad \| (\hat{U}_\epsilon - \kappa_0)|_{x=0,1} \|_{H^m(T \times \mathbb{R})} \leq 2M.
\]

Theorem 2.8 and Remark 3 guarantee that there exists a unique solution \( U_\epsilon \) to the following problem

\[
L(\hat{U}_\epsilon)U_\epsilon = F, \text{ in } Q, \quad B(\hat{U}_\epsilon)U_\epsilon = G, \text{ on } \partial Q,
\]

for all \( \gamma > \gamma_m \), where \( \gamma_m \) is independent of \( \epsilon \) (see (70) and Lemma 2.7), and \( U_\epsilon \) satisfies the following estimate

\[
\gamma \| U_\epsilon \|_{H^m(Q)}^2 + \epsilon_0 \| U_\epsilon |_{\partial Q} \|_{H^m(T \times \mathbb{R})}^2 \leq C(M) \left( \frac{1}{\gamma} \| F \|_{H^m(Q)}^2 + \| G \|_{H^m(T \times \mathbb{R})}^2 \right),
\]

for some generic constant \( C(M) > 0 \) independent of \( \epsilon \). The right-hand side of (72) being independent of \( \epsilon \), there exists \( U \in H^m(Q) \) and \( U_0 \in H^m(T \times \mathbb{R}) \) such that \( U_\epsilon \) (up to extracting a subsequence) converges weakly in \( H^m(Q) \) and strongly in \( e^{\gamma t}L^2(Q) \) to \( U \), while \( U_\epsilon |_{\partial Q} \) converges weakly in \( H^m(T \times \mathbb{R}) \) and strongly in \( e^{\gamma t}L^2(T \times \mathbb{R}) \) to \( U_0 \).

It remains to show that \( U |_{\partial Q} = U_0 \). Since \( L(\hat{U})(U_\epsilon - U) \) converges weakly to \( 0 \) in \( H^{m-1}(Q) \) and thus strongly in \( e^{\gamma t}L^2(Q) \), Proposition 2.1 implies that \( U_\epsilon |_{\partial Q} \) converges strongly to \( U |_{\partial Q} \) in \( e^{\gamma t}L^2(T \times \mathbb{R}) \), hence \( U_0 = U |_{\partial Q} \). By passing to the limit in (71), we see that \( U \) satisfies (2) and (5).

Finally, the \( H^m \)-estimate (68) follows for \( U \) by taking \( \text{lim inf} \) of the estimate (72), and the \( L^2 \)-estimate (66) follows with the same reasoning, by taking \( \text{lim inf} \) of the estimate (66) with \( U \) replaced by \( U_\epsilon \). The uniqueness follows by the energy estimate.

Lemma 2.10. Suppose that \( \hat{U} \) satisfies the conditions in Theorem 2.8. If \( F \in L^2(Q) \) and \( G \in L^2(T \times \mathbb{R}) \), and both vanish for \( t < t_0 \), then the solution \( U \) of the boundary value problem

\[
L(\hat{U})U = F, \text{ in } Q, \quad B(\hat{U})U = G, \text{ on } \partial Q
\]

also vanishes for \( t < t_0 \).

Remark 4. The proof of Lemma 2.10 follows the same arguments as the proof of [2, Theorem 9.13] with [2, Theorem 9.9] replaced by Theorem 2.8. Lemma 2.10 can be interpreted as the finite speed of propagation for hyperbolic systems.
Remark 5. Since the smoothness of $\hat{U}$ does not play any role in the proof of Lemma 2.10, this lemma is still valid if $\hat{U}$ satisfies the conditions in Theorem 2.9.

3. The linear shallow water system. In this section, we aim to study the linear shallow water system with homogeneous initial conditions, which prepares to study the fully nonlinear problem in the next section.

We consider the initial boundary value problem (IBVP) with zero initial data, that is
\[
\begin{cases}
L_G U = F, & \text{in } \Omega \times (0, T), \\
B(\tilde{U}) U = G, & \text{on } \mathbb{T} \times (0, T), \\
U(t = 0) = 0, & \text{in } \Omega.
\end{cases}
\]

(73)

In the following, we choose the constants $\kappa_0, \delta > 0$ such that
\[
\begin{align*}
&c_0 \leq \kappa_0 + c_3 \delta < c_1, \\
&(\kappa_0 + c_3 \delta)^2 - g(\kappa_0 - c_3 \delta) \leq -c_3^2,
\end{align*}
\]

where $c_3$ is given by the Lemma 3.1 below.

Lemma 3.1 (Extension Theorem). There exists a continuous linear operator $P = P_m$ from $H^m(\Omega \times [0, T])$ into $H^m(\Omega)$ such that for all $u \in H^m(\Omega \times [0, T])$, the restriction of $Pu$ to $\Omega \times [0, T]$ is $u$ itself, i.e.
\[
Pu|_{\Omega \times [0, T]} = u,
\]

and furthermore $Pu$ has compact support in the $t$-direction (i.e. in $\Omega$) and satisfies the estimate
\[
\|Pu\|_{L^\infty(\Omega)} \leq c_3 \|u\|_{L^\infty(\Omega \times [0, T])},
\]

\[
\|Pu\|_{H^m(\Omega)} \leq c_4 \|u\|_{H^m(\Omega \times [0, T])},
\]

where $c_3, c_4 > 1$ only depend on $m$, and are independent of $u$.

See [14, Chapter 2] for a detailed proof of Lemma 3.1, and the $L^\infty$-estimate comes from the reflection formula (4.8) in [14, Chapter 2]. See also [1, 8].

Theorem 3.2. Let $m \geq 3$ be an integer. Assume that $\hat{U}$ satisfies the following conditions:
\[
\|\hat{U} - \kappa_0\|_{L^\infty(\Omega \times [0, T])} \leq \delta, \|\hat{U} - \kappa_0\|_{H^m(\Omega \times [0, T])} \leq M, \|\hat{U} - \kappa_0\|_{H^m(\Omega \times [0, T])} \leq M.
\]

If $F \in H^m(\Omega \times [0, T])$ and $G \in H^m(\mathbb{T} \times [0, T])$ are such that $\partial_j^2 F = 0, \partial_j^2 G = 0$ at $t = 0$ for all $j \in \{0, \ldots, m - 1\}$, then there exists a unique solution $U$ to the IBVP (73), and the solution $U$ is in $H^m$, as well as its trace on the boundary $\{x = 0, 1\}$, and $U$ satisfies $\partial_j^2 U = 0$ at $t = 0$ for all $j \in \{0, \ldots, m - 1\}$, together with the $H^m$-energy estimate
\[
\frac{1}{T} \|U\|_{H^2(\Omega \times [0, T])} + \|U\|_{H^1(\mathbb{T} \times [0, T])} \leq C(M) (T \|F\|_{H^m(\Omega \times [0, T])} + \|G\|_{H^m(\mathbb{T} \times [0, T])}),
\]

(76)

and also the $L^2$-estimate
\[
\frac{1}{T} \|U\|_{L^2(\Omega \times [0, T])} + \|U\|_{L^2(\mathbb{T} \times [0, T])} \leq C(M) (T \|F\|_{L^2(\Omega \times [0, T])} + \|G\|_{L^2(\mathbb{T} \times [0, T])}),
\]

(77)

where $C(M) > 0$ depends on $M$, the $H^m$-norm of $\hat{U} - \kappa_0$ and other data, but is independent of $U$. 

Theorem 2.9, \( \tilde{U} \) is a solution of the homogeneous IBVP (73), and it belongs to \( H^m \).

The precaution (74) ensures that \( \tilde{U}^0 \) satisfies the subcritical condition (3).

The assumptions on \( F \) and \( G \) also allow us to extend \( F \) and \( G \), respectively, into \( \tilde{F} \in H^m_{\gamma} (Q) \) and \( \tilde{G} \in H^m_{\gamma} (\mathbb{T} \times \mathbb{R}) \), and both vanishing for \( t < 0 \). By Theorem 2.9, the corresponding BVP admits a unique solution \( \tilde{U} \in H^m_{\gamma} (Q) \), whose traces \( \tilde{U}|_{x=0,1} \) are in \( H^m_{\gamma} (\mathbb{T} \times \mathbb{R}) \). Furthermore, by Lemma 2.10 and Remark 5, \( \tilde{U} \) vanishes for \( t < 0 \), (and \( \tilde{U}|_{t \leq T} \) does not depend on \( \tilde{F}|_{t > T} \) and \( \tilde{G}|_{t > T} \)). Therefore, \( U := \tilde{U}|_{t \in [0,T]} \) is a solution of the homogeneous IBVP (73), and it belongs to \( H^m \), as well as its traces. By Theorem 2.9, \( \tilde{U} \) satisfies the estimate

\[
\gamma \| \tilde{U} \|^2_{H^k_\gamma (\Omega \times (-\infty,T])} + \epsilon_0 \| \tilde{U}|_{x=0,1} \|^2_{L^2_{\gamma} (\mathbb{T} \times (-\infty,T])} \leq C(M) \left( \frac{1}{\gamma} \| \tilde{F} \|^2_{H^k_\gamma (\Omega \times (-\infty,T])} + \| \tilde{G} \|^2_{H^k_\gamma (\mathbb{T} \times (-\infty,T])} \right),
\]

where \( k \) can be either \( m \) or 0. Using the fact that \( \tilde{U}, \tilde{F}, \tilde{G} \) vanish for \( t < 0 \) and choosing \( \gamma = 1/T \) in (78), we then obtain the \( H^m \)-estimate (76) and the \( L^2 \)-estimate (77).

It remains to show the uniqueness of the solution \( U \). We only show that the only \( L^2 \) solution of the homogeneous problem

\[
\begin{cases}
L_{\tilde{G}} U = 0, & \text{in } \Omega \times (0,T), \\
B(U) U = 0, & \text{on } \mathbb{T} \times (0,T), \\
U(t=0) = 0, & \text{in } \Omega.
\end{cases}
\]

is the trivial solution 0. The uniqueness for \( H^m \) solutions then follows. So we assume \( U \in L^2(\Omega \times (0,T)) \) is a solution of (79), and we denote by \( 1 \) the characteristic function of \( \mathbb{R}^+ \) in \( \mathbb{R} \), and set \( U^0(x,y,t) = U(x,y,t)1(t) \). We then see that \( U^0 \) solves the BVP

\[
\begin{cases}
L_{\tilde{G}} U = 0, & \text{in } \Omega \times (-\infty,T), \\
B(U^0) U = 0, & \text{on } \mathbb{T} \times (-\infty,T).
\end{cases}
\]

Now, we introduce a smooth cut-off function \( \theta(t) \) such that \( \theta = 1 \) on \((-\infty, \tau] \) with \( \tau < T \), and \( \theta = 0 \) on \([T, +\infty) \). Then both \( \theta U^0 \) and \( L_{\tilde{G}} (\theta U^0) \) belong to \( L^2(\Omega) \), and the traces \( B(U^0)(\theta U^0)|_{x=0,1} \) belong to \( L^2(\mathbb{T} \times \mathbb{R}) \). Furthermore, \( L_{\tilde{G}} (\theta U^0)|_{x=0,1} = 0 \) for \( t < \tau \). Hence by Lemma 2.10, \( \theta U^0 = 0 \) for \( t < \tau \). Since this is true for all \( \tau < T \) and \( \theta U^0 = U \) for \( t \in [0, \tau] \), we infer that \( U = 0 \) (a.e.) in \([0, T) \), as expected. This completes the proof of Theorem 3.2. \( \square \)

4. The fully nonlinear shallow water system. In this section, we aim to investigate the well-posedness for Eqs. (1) associated with suitable initial and boundary conditions. The fully nonlinear shallow water system reads in compact form

\[
U_t + \mathcal{E}_1(U)U_x + \mathcal{E}_2(U)U_y + \ell(U) = 0.
\]
4.1. **Stationary solutions.** We want to study system (80) near a stationary solution, and we start by constructing this stationary solution \((u, v, \phi) = (u_s, v_s, \phi_s)\). These functions are independent of time and satisfy

\[
E_1(U)U_x + E_2(U)U_y + \ell(U) = 0. 
\]  
(81)

In the following, we construct a \(y\)-independent stationary solution \(U_s\) of (81) satisfying the subcritical condition (3). Thus \(U_s\) satisfies (see Subsec. 2.1 in [9] for a stationary solution in the supercritical case):

\[
\begin{align*}
uu_x + g\phi_x - fv &= 0, \\
v_x + fu &= 0, \\
(u\phi)_x &= 0.
\end{align*}
\]  
(82)

We infer from (82) that

\[
\begin{align*}
u\phi &= \kappa_1, \\
v &= -fx + \kappa_2, \\
u^2 + 2g\phi &= -f^2x^2 + 2f\kappa_2x + \kappa_3,
\end{align*}
\]

where \(\kappa_1, \kappa_2, \kappa_3\) are constants. We first choose \(\kappa_1 = 1, \kappa_2 = 1\), and then we have \(\phi = u^{-1}, v = 1 - fx\) and

\[
u^2 + \frac{2g}{u} = -f^2x^2 + 2fx + \kappa_3, \quad x \in (0, 1).
\]  
(83)

We then set \(\Psi(u) = u^2 + \frac{2g}{u}\), and if we choose \(\kappa_3\) such that

\[
\Psi\left(\frac{1}{2g}\right) + f^2 < \kappa_3 < \Psi\left(\frac{1}{g}\right) + 2f,
\]

then for any \(x \in (0, 1)\) one solution (in \(u\)) of (83) is between \(1/2g\) and \(1/g\). We choose such a solution \(u\), and therefore \(\phi = u^{-1}\) satisfies \(g \leq \phi \leq 2g\), and furthermore \(u^2 - g\phi < 1/2g^2 - g^2 < 0\). Note that \(v\) is greater than \(1 - f > 0\) since the Coriolis parameter \(f \ll 1\). All these calculations mean that we can choose our stationary solution \(u_s, v_s, \phi_s\) satisfying the subcritical condition \(u^2 - g\phi < 0\) and \(u, v, \phi > 0\). Recall that we choose \(\kappa_0, \delta > 0\) such that

\[
\begin{align*}
c_0 \leq \kappa_0 &\pm c_3\delta < c_1, \\
(c_0 + c_3\delta)^2 - g(c_0 - c_3\delta) &\leq -c_2^2,
\end{align*}
\]  
(84)

where \(c_0, c_1, c_2\) are given positive constants, and \(c_3\) is given by Lemma 3.1.

In what follows, we think of the stationary solution \(U_s\) in a general form (i.e. \(U_s\) depends both on \(x\) and \(y\)), and we choose \(U_s = (u_s, v_s, \phi_s)\) such that

\[
|U_s - \kappa_0| \leq \delta/4,
\]  
(85)

and by (84), \(U_s\) satisfies the subcritical condition (3).

We set \(U = U_s + \tilde{U}\) and substitute these values into (80); we obtain a new system for \(\tilde{U}\), and dropping the tildes, our new system reads:

\[
U_t + E_1(U + U_s)U_x + E_2(U + U_s)U_y + \ell(U) = -E_1(U + U_s)U_s,x - E_2(U + U_s)U_s,y - \ell(U_s).
\]  
(86)
We supplement (86) with the following initial and boundary conditions:

\[
\text{I.C. } U(0) = U_0, \quad \text{B.C. } \begin{cases} 
  u + u_s + 2\sqrt{g(\phi + \phi_s)} = g_1, & \text{on } x = 0, \\
  v + v_s = g_2, & \text{on } x = 0, \\
  u + u_s - 2\sqrt{g(\phi + \phi_s)} = g_3, & \text{on } x = 1.
\end{cases}
\] (87)

We regard the initial condition \(U_0\) as a small perturbation of the stationary solution, and we choose \(U_0\) satisfying

\[
|U_0| \leq \delta/4.
\] (88)

For simplicity, we set

\[
b(U_s + U) = \begin{pmatrix} (u + u_s + 2\sqrt{g(\phi + \phi_s)})_{|x=0} \\
  (v + v_s)_{|x=0} \\
  (u + u_s - 2\sqrt{g(\phi + \phi_s)})_{|x=1}
\end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\
  g_2 \\
  g_3
\end{pmatrix},
\]

and we have \(b(U_s + U) = G\) representing the boundary conditions in (87).

The disadvantage of this new formulation (86)-(87) is that the initial condition \(U_0\) is non-zero. To overcome this difficulty, we use \(U_a = (u_a, v_a, \phi_a)\) an approximate lifting of the initial condition \(U_0\), which is given by Lemma 4.1 below, and if we have

\[
|U_a - U_0| \leq \delta/4, \quad |U| \leq \delta/4,
\] (89)

then

\[
|U_a + U_s + U - \kappa_0| \leq \delta.
\]

Hence by (84), we have

\[
\begin{cases} 
  c_0 \leq U_a + U_s + U \leq c_1, \\
  (u_a + u + u_s)^2 - g(\phi_a + \phi + \phi_s) \geq -c_2^2.
\end{cases}
\] (90)

The estimates (90) will guarantee that we remain in the subcritical case.

4.2. Conditions on the data. In order to be able to solve system (86) we need to introduce some technical conditions (see [2, Section 11.1.2]). First we require that \(U = 0\) is a solution of the special IBVP (86) with zero initial data and boundary data \(G(\cdot, t = 0)\). In view of that \(U_s\) is a stationary solution satisfying (81), we require the compatibility conditions:

\[
G(\cdot, t = 0) = b(U_s).
\] (91)

The second condition is that the initial and boundary data should satisfy some additional compatibility conditions and these conditions are very natural for smooth solutions, which we are looking for. Let us rewrite (86) as

\[
U_t = H(U + U_s) - \mathcal{E}_1(U + U_s)U_x - \mathcal{E}_2(U + U_s)U_y - \ell(U),
\] (92)

where we denote by \(H(U + U_s)\) the right-hand side of (86). Now, if \(U\) is smooth up to the boundary, then necessarily

\[
b(U_s + U_0) = G(y, 0);
\] (93)

and if \(U\) is \(C^1\) up to the boundary,

\[
\partial_t G(y, 0) = db(U_s + U_0) \cdot \partial_t U(x, 0)
\]

\[
= db(U_s + U_0) \cdot (H(U_0 + U_s) - \mathcal{E}_1(U_0 + U_s)U_{0,x} - \mathcal{E}_2(U_0 + U_s)U_{0,y} - \ell(U_0)).
\]

More general, if \(U\) is \(C^{m-1}\) up to the boundary, then

\[
\partial^p \ell G(y, 0) = \mathcal{C}_{p,U_0}(V_0, \cdots, V_p), \quad \forall p \in \{1, \cdots, m-1\},
\] (94)
where the complicated nonlinear function $C_{p, \nu_0}$ is
\[
C_{p, \nu_0}(V_0, \cdots, V_p) = \sum_{k=1}^{p} \sum_{j_1+\cdots+j_m=p} c_{j_1, \cdots, j_k} d^m b(U_s + U_0) \cdot (V_{j_1}, \cdots, V_{j_k}),
\]
and the functions $V_i$ ($i = 0, \cdots, m$) are defined by induction
\[
V_0 = U,
V_1 = \partial_t U = H(U + U_s) - \mathcal{E}_1(U + U_s)U_x - \mathcal{E}_2(U + U_s)U_y - \ell(U),
\]
and for all $i = 1, \cdots, m - 1$,
\[
V_{i+1} = \partial_t^{i+1} U = \sum_{k=1}^{i} \sum_{j_1+\cdots+j_k=i} c_{j_1, \cdots, j_k} (d^k H(U + U_s)) \cdot (V_{j_1}, \cdots, V_{j_k}),
\]
\[
- \sum_{l=1}^{i} \left( \sum_{k=1}^{l} \sum_{j_1+\cdots+j_k=l} c_{j_1, \cdots, j_k} (d^k \mathcal{E}_1(U + U_s)) \cdot (V_{j_1}, \cdots, V_{j_k}) V_{i-l,x}
\right)
\]
\[
- \sum_{l=1}^{i} \left( \sum_{k=1}^{l} \sum_{j_1+\cdots+j_k=l} c_{j_1, \cdots, j_k} (d^k \mathcal{E}_2(U + U_s)) \cdot (V_{j_1}, \cdots, V_{j_k}) V_{i-l,y}
\right)
- \mathcal{E}_1(U + U_s)V_{i,x} - \mathcal{E}_2(U + U_s)V_{i,y} - \ell(V_i),
\]
where the coefficients $c_{j_1, \cdots, j_k}$ are derived from the Faà di Bruno’s formula, see \cite{3, 5}. The conditions (93)-(94) express the classical compatibility conditions which are necessary for the solution $U$ of (86) to be $C^{m-1}$ near $t = 0$; see e.g. \cite{19, 21, 24}.

4.3. Approximate solutions. Approximate solutions and related estimates are obtained by the following lemmas (for details see \cite{2}, Chapter 11):

**Lemma 4.1.** Given $U_0 = (u_0, v_0, \phi_0) \in H^{m+1/2}(\Omega)$ with $m \geq 3$, there exists $T_0 > 0$ and $U_a \in H^{m+1}(\Omega \times \mathbb{R})$, vanishing for $|t| \geq 2T_0$ and such that $U_{a|t=0} = U_0$,
\[
|U_a - U_0| \leq \delta/4, \text{ for any } (x, y, t) \in \Omega \times [-T_0, T_0].
\]
If we let $F^0 = -L_{U_a+U_s}(U_a + U_s)$ and $G^0 = -b(U_a + U_s) + G$, then $F^0 \in H^m(\Omega \times \mathbb{R}), G^0 \in H^m(\mathbb{T} \times \mathbb{R})$, and
\[
\partial_t^j F^0 = 0, \partial_t^j G^0 = 0, \text{ on } t = 0, \text{ for any } j \in \{0, \cdots, m-1\}.
\]

This lemma provides a lifting of the initial data $U_0$ by a function $U_a$ which yields the flatness properties (96).

In what follows we shall denote by $I_T$ the half-line time interval $(-\infty, T)$, and $\Omega_T$ (resp. $T_T$) is $\Omega \times I_T$ (resp. $\mathbb{T} \times I_T$). More generally we introduce the notations
\[
F^U = -L_{U_a+U_s+U}(U_a + U_s), \quad G^U = B(U_a + U_s + U)U - b(U_a + U_s + U) + G.
\]

**Lemma 4.2.** Assume that $T \in (0, T_0]$, and $U \in H^m(\Omega_T)$ with $m \geq 3$ such that
\[
\|U\|_{H^m(\Omega_T)} \leq M \in (0, \frac{\delta}{4\nu_m}], \quad \|U\|_{x=0, t=0} \leq M, \quad U|_{t=0} = 0,
\]
where $\nu_m$ denotes the norm of the Sobolev embedding $H^m(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Then we have
\[
\partial_t^j F^U = 0, \partial_t^j G^U = 0, \text{ at } t = 0, \text{ for any } j \in \{0, \cdots, m-1\},
\]
and
\[
\|F^U\|_{H^m(\Omega_T)} \leq C(M), \quad \|G^U\|_{H^m(\mathbb{T}_T)} \leq TC(M) + \epsilon(T),
\]
where $\epsilon(T)$ is independent of $M$ and goes to zero as $T$ goes to zero.
Assume that the scheme defined as follows:

Based on the observation (99), we now introduce our iterative constructive proof.

In Lemma 4.2, the $L^\infty(\Omega_T)$-norm of $U$ is less than $M$; then by the Sobolev embedding, the $L^\infty(\Omega_T)$ norm of $U$ is less than $\frac{\delta}{4}$, and together with (95) in Lemma 4.1, we see that this $U$ will stay in our admissible set (see (89)-(90)).

4.4. Iteration scheme. We recall that we set $U = U_s + \tilde{U}$ and that we dropped the tilde in the above. Now let us reintroduce the tilde and set $\tilde{U} = U_a + \tilde{U}$, so that $U = U_s + U_a + \tilde{U}$. We then substitute it into system (86), drop the bar and use the notation $F_U$; therefore the new system for $U = \tilde{U}$ becomes:

\[
\begin{cases}
L_{U_a+U_s}U = -L_{U_a+U_s}(U_a + U_s) = F_U, \\
U|_{t=0} = 0, \\
b(U_a + U_s + U) = G.
\end{cases}
\]  

(98)

Applying $\partial_t^j$ to the first equation (98), we find

\[
\partial_t^{j+1}U + \partial_t^j(E_1(U_a + U + U_s) + E_2(U_a + U + U_s)U_y) = \partial_t^j F_U,
\]

which is equivalent to

\[
\partial_t^{j+1}U = \partial_t^j F_U - \sum_{0 \leq i \leq j} (\partial_t^{j-i}E_1(U_a + U + U_s)\partial_t^i U_x + \partial_t^{j-i}E_2(U_a + U + U_s)\partial_t^i U_y).
\]

By Lemma 4.2, we have

\[
\partial_t^j F_U|_{t=0} = 0, \quad \forall \ j \in \{0, \cdots, m - 1\},
\]

and we also have $U|_{t=0} = 0$; so by induction, we see that

\[
\partial_t^j U|_{t=0} = 0, \quad \forall \ j \in \{0, \cdots, m\}.
\]  

(99)

Hence we can extend $U$ by zero for $t < 0$ in $H^m(\Omega_T)$, and we see that the first equation (98) is valid for $(x, y, t) \in \Omega_T$, the second equation (98) is valid for $(x, y, t) \in \Omega \times (-\infty, 0]$, and the third equation (98) is valid for $(y, t) \in \mathbb{T}_T$.

We are now on the stage to prove the main result.

Theorem 4.3 (For the nonlinear system (98)). Let $m \geq 3$ be fixed and let there be given $U_0 = (u_0, v_0, h_0)$ in $H^{m+\frac{3}{2}}(\Omega)$ satisfying the condition (88), and $G(y, t) = (g_u(y, t), g_v(y, t), g_h(y, t))$ in $H^m(\mathbb{T}_T)$ such that $G(y, t = 0) = 0$ (see (91)) and satisfies the compatibility conditions (93)-(94). Then there exists $T_* > 0$ depending on the initial and boundary data and also on the stationary solution $U_s$ such that the system (98) admits a unique solution $U = (u, v, \phi)$ in $\Omega \times (0, T_*)$ satisfying:

\[
U \in H^m(\Omega \times (0, T_*)).
\]  

(100)

Consequently, the system (86) admits a unique solution $U_a + U \in H^m(\Omega \times (0, T_*)).$

Proof. Based on the observation (99), we now introduce our iterative constructive scheme defined as follows:

\[
\begin{cases}
L_{U_a+U_s}U^{k+1} = F_U^k, \quad (x, y, t) \in \Omega_T, \\
U^{k+1}|_{t=0} = 0, \\
B(U_a + U_s + U^k)U^{k+1} = G_U^k, \quad (y, t) \in \mathbb{T}_T.
\end{cases}
\]  

(101)

Construction of the sequence $\{U^k\}$: We fix $M \in (0, \frac{\delta}{4m})$. We initiate our iteration scheme (101) by setting $U^0 = 0$, and construct the $U^k$ by induction. Assume that $U^k$ has been constructed in such a way that for some $T \in (0, T_0]$,

\[
||U^k||_{H^m(\Omega_T)} \leq M, \quad ||U^k||_{H^m(\mathbb{T}_T)} \leq M, \quad U^k|_{t \leq 0} = 0.
\]  

(102)
By Lemma 4.2, we have that both $F^{U_k}$ and $G^{U_k}$ satisfy (97). Set $\hat{U}_k = U_a + U_s + U^k$, then $\hat{U}_k - \kappa_0$ belongs to $H^m$ as well as its traces $(\hat{U} - \kappa_0)|_{x=0,1}$, and $|\hat{U}_k - \kappa_0| \leq \delta$ (see Remark 6).

Then by Theorem 3.2 the IBVP (101) admits a unique solution $U^{k+1}$ satisfying $U^{k+1}|_{t=0} = 0$ and enjoying the estimate

$$
\frac{1}{T} \|U^{k+1}\|_{H^m(\Omega_T)}^2 + c_0 \|U^{k+1}\|_{C^1(\Omega_T)}^2 + \|G^{U_k}\|_{H^m(\Omega_T)}^2 + \|F^{U_k}\|_{H^m(\Omega_T)}^2 \leq C(M)(T\|F^{U_k}\|_{H^m(\Omega_T)} + \|G^{U_k}\|_{H^m(\Omega_T)}),
$$

where $C(M)$ only depends on the bound of the $H^m$-norm of $\hat{U}_k - \kappa_0$, i.e., only depends on $M$ since $U_a, U_s, \kappa_0$ are fixed in the iteration scheme. By virtue of Lemma 4.2, (103) implies that

$$
\frac{1}{T} \|U^{k+1}\|_{H^m(\Omega_T)}^2 + c_0 \|U^{k+1}\|_{C^1(\Omega_T)}^2 \leq C(M)(T\|F^{U_k}\|_{H^m(\Omega_T)} + \|G^{U_k}\|_{H^m(\Omega_T)} + e(T)),
$$

of which the right-hand side approaches 0 when $T \to 0$. We thus can choose $T$ small enough such that

$$
\|U^{k+1}\|_{H^m(\Omega_T)} \leq M, \quad \|U^{k+1}\|_{C^1(\Omega_T)} \leq M, \quad U^{k+1}|_{t=0} = 0.
$$

Now $U^{k+1}$ also satisfies (102), hence we can continue our construction. Let us emphasize that the choice of $T$ only depends on $M$ and is independent of $k$, therefore our iteration scheme can be conducted for all $k$, and we can construct the sequence $\{U^k\}$ since the starting point $U^0 = 0$ clearly satisfies the conditions (102).

Convergence of the sequence $\{U^k\}$: From the uniform $H^m$-bound in (102), we already know that there exists a subsequence of $\{U^k\}$ converging weakly in $H^m$. The next point is to prove that the sequence $\{U^k\}$ is Cauchy in $L^2$. In this respect, we write $W^{k+1} = U^{k+1} - U^k$, and then by direct computations, we obtain that $W^{k+1}$ satisfies

$$
\begin{cases}
L_{\hat{U}_k} W^{k+1} = (L_{\hat{U}_k} - L_{\hat{U}_k}) \hat{U}_k, \\
W^{k+1}|_{t=0} = 0,
\end{cases}
$$

Hence, by the $L^2$-estimate (77), we obtain

$$
\frac{1}{T} \|W^{k+1}\|_{L^2(\Omega_T)}^2 + c_0 \|W^{k+1}\|_{C^1(\Omega_T)}^2 \leq C(M)(T\|L_{\hat{U}_k} - L_{\hat{U}_k}\|_{L^2(\Omega_T)}^2 + \|B(\hat{U}_k)W^k + b(\hat{U}_k) - b(\hat{U}_k)\|_{L^2(\Omega_T)}^2).
$$

Merely by the mean value theorem we obtain that

$$
\|(L_{\hat{U}_k} - L_{\hat{U}_k}) \hat{U}_k\|_{L^2(\Omega_T)} \leq C\|\hat{U}_k\|_{H^m(\Omega_T)} \|W^k\|_{L^2(\Omega_T)} \leq C(M)\|W^k\|_{L^2(\Omega_T)},
$$

where we used the inequality $\|\hat{U}_k\|_{W^{1,\infty}(\Omega_T)} \lesssim \|\hat{U}_k\|_{H^m(\Omega_T)}$ for $m \geq 3$.

By a second-order Taylor expansion of $b$ and noticing that the derivative $db = B$, direct computation yields that

$$
|B(\hat{U}_k^{k-1})W^k + b(\hat{U}_k^{k-1}) - b(\hat{U}_k)| \leq C|W^k|^2,
$$
where $C$ is a constant depending on the lower bound $c_0$ of $\phi_n + \phi_s + \phi^k$, independent of $k$. Then we obtain

$$
\|B(\tilde{U}^{k-1})W^k + b(\tilde{U}^{k-1}) - b(\tilde{U}^{k})\|_{L^2(T^\gamma)} \leq C\|W^k\|_{L^\infty(T^\gamma)}\|W^k\|_{L^2(T^\gamma)}. \tag{108}
$$

Now by the Sobolev embedding again:

$$
\|W^k\|_{L^\infty(T^\gamma)}^2 \lesssim \|W^k\|_{H^{m}(T^\gamma)}^2 \leq \frac{2C(M)}{\epsilon_0}(TC(M) + TC(M) + \epsilon(T)), \tag{109}
$$

thanks to the $H^m$-estimate (104). Therefore, gathering all the estimates (106)-(109), we finally arrive at

$$
\frac{1}{T}\|W^{k+1}\|^2_{L^2(\Omega_T)} + \epsilon_0\|W^{k+1}\|_{L^2(T^\gamma)} \leq TC(M)\|W^{k}\|^2_{L^2(\Omega_T)} + \epsilon_M(T)\|W^{k}\|^2_{L^2(T^\gamma)}, \tag{110}
$$

where $\epsilon_M(T)$ goes to zero as $T$ goes to zero. Consequently, up to diminishing $T$ so that all four numbers

$$
C(M)T^2, \epsilon_M(T)T, \epsilon_M(T)/\epsilon_0, C(M)T/\epsilon_0,
$$

are less than $1/4$, we obtain

$$
\begin{align*}
\|W^{k+1}\|_{L^2(\Omega_T)} &\leq 2^{-k}(\|W^1\|^2_{L^2(\Omega_T)} + \|W^1\|^2_{L^2(T^\gamma)}), \\
\|W^{k+1}\|_{L^2(T^\gamma)} &\leq 2^{-k}(\|W^1\|^2_{L^2(\Omega_T)} + \|W^1\|^2_{L^2(T^\gamma)}),
\end{align*} \tag{111}
$$

which shows that both $\{U^k\}$ and $\{U^k\}_{x=0,1}$ are Cauchy sequences in $L^2$. Let us call $U$ and $U$ the limits. Noting that the $H^m$-norms of $\{U^k\}$ and $\{U^k\}_{x=0,1}$ are uniformly bounded, hence, by the $L^2 - H^m$ interpolation, $\{U^k\}$ and $\{U^k\}_{x=0,1}$ converge strongly to $U$ and $U$ in $H^{m-1}$, respectively. Therefore, $\hat{U} = U|_{x=0,1}$, and $U$ solves the IBVP (98), and $U_x + U$ solves the IBVP (86).

It only remains to prove the uniqueness of the solution. This follows immediately from (110) by taking $\hat{U}$ the difference between two solutions satisfying the same initial and boundary conditions. We obtain $\|U\|_{L^2(\Omega_T)} = 0$ which enables us to conclude that the solution is unique. This completes the proof of Theorem 4.3. \qed

Appendix A. The sobolev spaces. In this appendix, we first give some results about the Sobolev spaces on the torus. The idea of the proof for these results already exists in Chapter 2 and Chapter 4 of [4], where they consider the Sobolev spaces. The idea of the proof for these results already exists in Chapter 2 and Chapter 4 of [4], where they consider the Sobolev spaces on the whole space. We then extend these results to the product space of the torus and the whole space.

A.1. The $n$-Torus $T^n$. The $n$-torus $T^n$ is the cube $[0, 1]^n$ with opposite sides identified. A more precise definition can be given as follows: For $x, y$ in $\mathbb{R}^n$, we say that

$$
x \equiv y
$$

if $x - y \in \mathbb{Z}^n$. Here $\mathbb{Z}^n$ is the additive subgroup of all points in $\mathbb{R}^n$ with integer coordinates. It is a simple fact that $\equiv$ is an equivalence relation that partitions $\mathbb{R}^n$ into equivalence classes. The $n$-torus $T^n$ is then defined as the set $\mathbb{R}^n/\mathbb{Z}^n$ of all such equivalence classes.

A.2. Fourier coefficients.

Definition A.1. For $f \in L^1(\mathbb{T}^n)$ and $\xi \in \mathbb{Z}^n$, we define

$$\hat{f}(\xi) = \int_{\mathbb{T}^n} f(x)e^{-2\pi i \xi \cdot x} dx. \quad (112)$$

We call $\hat{f}(\xi)$ the $\xi$-th Fourier coefficient of $f$.

Note that the Fourier transform $f \mapsto \hat{f}$ maps $L^1(\mathbb{T}^n)$ into $l^\infty(\mathbb{Z}^n)$. If $f$ is sufficiently smooth, then integration by parts yields

$$\partial^\alpha f(\xi) = (2\pi i)^\alpha \hat{f}(\xi), \quad x^\alpha \hat{f}(\xi) = \left(\frac{i}{2\pi}\right)^\alpha \partial^\alpha \hat{f}(\xi). \quad (113)$$

Using Fubini’s theorem, we can also obtain

$$\hat{f} * g(\xi) = \hat{f}(\xi) \hat{g}(\xi). \quad (114)$$

We also have Plancherel’s identity

$$\|f\|_{L^2(\mathbb{T}^n)}^2 = \sum_{\xi \in \mathbb{Z}^n} |\hat{f}(\xi)|^2, \quad \forall f \in L^2(\mathbb{T}^n). \quad (115)$$

For more properties about the Fourier coefficients, see e.g. [7, Chapter 3].

A.3. Sobolev spaces $H^s(\mathbb{T}^n)$. Let $\mathcal{S}'(\mathbb{T}^n)$ be the dual space of $\mathcal{S}(\mathbb{T}^n)$ which includes all infinitely differentiable periodic functions on $\mathbb{T}^n$, and the space $\mathcal{S}'(\mathbb{T}^n)$ is called the space of periodic distributions. For all $s \in \mathbb{R}$, the Hilbert spaces $H^s(\mathbb{T}^n)$ are defined as follows:

$$H^s(\mathbb{T}^n) = \{ f \in \mathcal{S}'(\mathbb{T}^n) \mid \|f\|_{H^s(\mathbb{T}^n)} < \infty \},$$

where the norm on $H^s(\mathbb{T}^n)$ is defined as

$$\|f\|_{H^s(\mathbb{T}^n)}^2 = \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2, \quad (116)$$

with $|\xi|^2 = \sum_{i=1}^n \xi_i^2$.

Proposition A.2. If $f \in H^s(\mathbb{T}^n)$ and if there exists $C > 0$ such that

$$\|f\|_{H^{s,\theta}}^2 := \sum_{\xi \in \mathbb{Z}^n} \frac{(1 + |\xi|^2)^{s+1}}{1 + |\theta \xi|^2} |\hat{f}(\xi)|^2 \leq C,$$

for all $\theta \in (0, 1]$, then $f$ belongs to $H^{s+1}(\mathbb{T}^n)$.

Proposition A.2 is a consequence of Fatou’s Lemma: take the lim inf of the inequality when $\theta$ goes to 0. So, showing additional regularity for $f \in H^s(\mathbb{T}^n)$ amounts to finding a uniform bound of $\|f\|_{s,\theta}$.

In order to find the uniform bound, we need to regularize the function and we thus introduce some mollifiers. We let $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(0) \neq 0$, where $\hat{\chi}$ denotes the Fourier transform of $\chi$ in $\mathbb{R}^n$, i.e.

$$\hat{\chi}(\xi) = \int_{\mathbb{R}^n} \chi(x)e^{-2\pi i \xi \cdot x} dx, \quad \forall \xi \in \mathbb{R}^n.$$

We define $\rho \in \mathcal{S}(\mathbb{R}^n)$ by setting

$$\hat{\rho}(\xi) = |\xi|^\gamma \hat{\chi}(\xi), \quad \gamma \in \mathbb{R}. \quad (117)$$
Then it is easy to verify that \( \rho \) satisfies the following two properties:

\[
\begin{aligned}
&\{\hat{\rho}(\xi) = O(|\xi|^n) \text{ in a neighborhood of 0 as } \hat{\chi}(0) \neq 0, \\
&\hat{\rho} \text{ does not vanish identically on any ray } \{t\xi : t \in \mathbb{R}^+\}, \xi \neq 0.
\end{aligned}
\]  

(118)

The second property can be verified by contradiction and using \( \hat{\chi}(0) \neq 0 \).

For any \( 0 < \epsilon \leq 1 \), we set \( \rho_\epsilon = \epsilon^{-n} \rho(x/\epsilon) \). We now define a version of \( \rho_\epsilon \) on \( \mathbb{T}^n \) by setting

\[
\sigma_\epsilon(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{\rho}(\epsilon\xi)e^{2\pi i \xi \cdot x}.
\]  

(119)

Then we have

\[
\hat{\sigma}_\epsilon(\xi) = \hat{\rho}_\epsilon(\xi) = \hat{\rho}(\epsilon\xi), \quad \forall \xi \in \mathbb{Z}^n.
\]  

(120)

**Proposition A.3.** Let \( s \in \mathbb{R}, \rho \) and \( \sigma_\epsilon \) defined by (117) and (119) with \( \gamma > s + 1 \). Then there exists \( C_1 \) and \( C_2 > 0 \) such that

\[
\begin{aligned}
&\|\sigma_\epsilon \ast f\|_{L^2_x(\mathbb{T}^n)}^2 \leq \|\sigma_\epsilon \ast f\|_{H^s(\mathbb{T}^n)}^2 + \mathcal{P}_{s,\theta}(f) \leq C_2\|f\|_{L^2_x(\mathbb{T}^n)}^2,
&\mathcal{P}_{s,\theta}(f) := \int_0^1 \|\sigma_\epsilon \ast f\|_{L^2_x(\mathbb{T}^n)}^2 e^{-2(s+1)}(1 + \frac{\theta^2}{\epsilon^2})^{-1}\frac{d\epsilon}{\epsilon},
\end{aligned}
\]  

(121)

for all \( \theta \in (0, 1) \) and all \( f \in H^s(\mathbb{T}^n) \).

**Proof.** We have by Plancherel’s identity and relation (120)

\[
\|\sigma_\epsilon \ast f\|_{L^2_x(\mathbb{T}^n)}^2 = \sum_{\xi \in \mathbb{Z}^n} |\sigma_\epsilon \ast f(\xi)|^2 = \sum_{\xi \in \mathbb{Z}^n} |\hat{\sigma}_\epsilon(\xi)\hat{f}(\xi)|^2 = \sum_{\xi \in \mathbb{Z}^n} |\hat{\rho}(\epsilon\xi)|^2|\hat{f}(\xi)|^2,
\]

which shows that we can write

\[
\mathcal{P}_{s,\theta}(f) = \int_0^1 \|\sigma_\epsilon \ast f\|_{L^2_x(\mathbb{T}^n)}^2 e^{-2(s+1)}(1 + \frac{\theta^2}{\epsilon^2})^{-1}\frac{d\epsilon}{\epsilon} = \sum_{\xi \in \mathbb{Z}^n} F(\xi, \theta)|\hat{f}(\xi)|^2,
\]

with

\[
F(\xi, \theta) = \int_0^1 |\hat{\rho}(\epsilon\xi)|^2 e^{-2(s+1)}(1 + \frac{\theta^2}{\epsilon^2})^{-1}\frac{d\epsilon}{\epsilon}.
\]  

(122)

In order to show (121), it is sufficient to show that there exists constants \( C_1 \) and \( C_2 > 0 \) such that

\[
C_1 \frac{(1 + |\xi|^2)^{s+1}}{1 + |\theta\xi|^2} \leq (1 + |\xi|^2)^s + F(\xi, \theta) \leq C_2 \frac{(1 + |\xi|^2)^{s+1}}{1 + |\theta\xi|^2}, \quad \forall \xi \in \mathbb{Z}^n, \theta \in (0, 1),
\]

which is equivalent to showing that

\[
C_1 \frac{|\xi|^{2(s+1)}}{(1 + |\theta\xi|^2)} \leq F(\xi, \theta) \leq C_2 \frac{|\xi|^{2(s+1)}}{(1 + |\theta\xi|^2)}, \quad \forall \xi \in \mathbb{Z}^n, \theta \in (0, 1).
\]  

(123)

It is clear that (123) holds for \( \xi = 0 \). We now assume that \( |\xi| \geq 1 \) since \( \xi \in \mathbb{Z}^n \). If we put \( \eta = \frac{\xi}{|\xi|} \) and \( t = \epsilon |\xi| \) in the integral (122), we obtain

\[
F(\xi, \theta) = |\xi|^{2(s+1)} \int_0^{\epsilon |\xi|} |\hat{\rho}(t\eta)|^2 t^{-2(s+1)}(1 + \frac{|\theta\xi|^2}{\epsilon^2})^{-1}\frac{dt}{t}.
\]

We therefore deduce

\[
F(\xi, \theta) \leq |\xi|^{2(s+1)}(1 + |\theta\xi|^2)^{-1} \left[ \int_0^{\epsilon |\xi|} |\hat{\rho}(t\eta)|^2 t^{-2(s+1)}\frac{dt}{t} + \int_1^{\infty} |\hat{\rho}(t\eta)|^2 t^{-2s-2} \frac{dt}{t} \right],
\]
Lemma A.4. Let the following lemma which gives a sort of limited expansion with respect to \( \epsilon \) for all \( x \) in the Schwartz space for the second integral, we thus obtain the second inequality in (123).

In order to prove the first inequality in (123), we observe that

\[
F(\xi, \theta) \geq \frac{1}{4} |\xi|^{2(s+1)} (1 + |\theta\xi|^{2})^{-1} \int_{1/2}^{1} |\hat{\rho}(t\eta)|^{2} t^{2(s+1)} \frac{dt}{t},
\]

and we take

\[
C_1 = \inf_{|\eta|=1} \frac{1}{4} \int_{1/2}^{1} |\hat{\rho}(t\eta)|^{2} t^{2(s+1)} \frac{dt}{t}.
\]

We have \( C_1 > 0 \), since \( \hat{\rho} \) does not vanish identically on any ray (see (118)\(_1\)). We thus completed the proof.

We now concern ourselves with studying the behavior of the commutator of a differential operator \( a(x)\partial^{\alpha} \) with a regularization \( \sigma_{\epsilon}* \); this study is based on the following lemma which gives a sort of limited expansion with respect to \( \epsilon \) of \([a(x)\partial^{\alpha}, \sigma_{\epsilon}*] \) when \( \epsilon \to 0 \).

**Lemma A.4.** Let \( \alpha \) be a multi-index with \( |\alpha| = m \), \( \rho \) and \( \sigma_{\epsilon} \) defined by (117) and (119), and \( a(x) \) belongs to \( \mathcal{C}^{\infty}(\mathbb{T}^{n}) \). We denote by \( \rho_{\epsilon}^{\beta} \) the function \( \epsilon^{-n}\rho^{\beta}(x/\epsilon) \) with \( \rho^{\beta}(x) = x^{\beta}\rho(x) \) and also set

\[
\sigma_{\epsilon}^{\beta}(x) := \sum_{\xi \in \mathbb{Z}^{n}} \hat{\rho}_{\epsilon}^{\beta}(\xi) e^{2\pi i \xi \cdot x} = \sum_{\xi \in \mathbb{Z}^{n}} \left( \frac{i}{2\pi} \right)^{\beta} \partial^{\beta} \hat{\rho}(\xi) e^{2\pi i \xi \cdot x},
\]

where \( \beta \) is also a multi-index. If we have \( s \in \mathbb{R} \) and \( \delta \geq 0 \), then for all \( N \in \mathbb{N} \), there exists \( C > 0 \) such that

\[
a\partial^{\alpha}(\sigma_{\epsilon} * u) - \sigma_{\epsilon} * (a\partial^{\alpha} u) = \sum_{0 < |\beta| < N} \frac{\epsilon^{|eta|}}{\beta!} \sigma_{\epsilon}^{\beta} * (\partial^{\beta} a\partial^{\alpha} u) + R_{N}(\epsilon) u,
\]

with

\[
||R_{N}(\epsilon) u||_{H^{s+\delta}(\mathbb{T}^{n})} \leq C \epsilon^{N-\delta} ||u||_{H^{s+m}(\mathbb{T}^{n})},
\]

for all \( u \in H^{s+m}(\mathbb{T}^{n}) \) and \( 0 < \epsilon \leq 1 \).

**Proof.** Using Fourier coefficients, we can write

\[
a(x)\partial^{\alpha} u(x) = \sum_{\xi, \eta \in \mathbb{Z}^{n}} \hat{a}(\xi - \eta)(2\pi i \eta)^{\alpha}\hat{u}(\eta) e^{2\pi i \xi \cdot x}, \quad \forall u \in \mathcal{C}^{\infty}(\mathbb{T}^{n});
\]

hence, for \( u \in \mathcal{C}^{\infty}(\mathbb{T}^{n}) \), using relation (120), we obtain

\[
a(x)\partial^{\alpha}(\sigma_{\epsilon} * u)(x) = \sum_{\xi, \eta \in \mathbb{Z}^{n}} \hat{a}(\xi - \eta)\hat{\rho}(\epsilon \eta)(2\pi i \eta)^{\alpha}\hat{u}(\eta) e^{2\pi i \xi \cdot x},
\]

and in a similar fashion, we obtain

\[
\sigma_{\epsilon} * (a\partial^{\alpha} u)(x) = \sum_{\xi, \eta \in \mathbb{Z}^{n}} \hat{a}(\xi - \eta)\hat{\rho}(\epsilon \xi)(2\pi i \xi)^{\alpha}\hat{u}(\eta) e^{2\pi i \xi \cdot x}.
\]
We also rewrite the summand in the right-hand side of (124) as
\[
\sum_{\xi, \eta \in \mathbb{Z}^n} e^{i\frac{|\beta|}{\beta!} (\delta^\beta a \partial^\alpha) v(x)}
\]
\[
= \sum_{\xi, \eta \in \mathbb{Z}^n} e^{i\frac{|\beta|}{\beta!} (\frac{i}{2\pi})^\beta \partial^\beta \tilde{\rho}(\epsilon \xi - \eta) (2\pi i (\xi - \eta))} \partial^\beta \tilde{\rho}(\epsilon \xi - \eta) (2\pi i) \alpha \partial^\nu \tilde{u}(\eta) e^{2\pi i \xi \cdot x}
\]
(128)
\[
= \sum_{\xi, \eta \in \mathbb{Z}^n} e^{i\frac{|\beta|}{\beta!} (\xi - \eta)^\beta \partial^\beta \tilde{\rho}(\epsilon \xi - \eta) (2\pi i) \alpha \partial^\nu \tilde{u}(\eta) e^{2\pi i \xi \cdot x}.
\]
We utilize the Taylor formula up to order \(N\) to expand with respect to \(\epsilon\) the difference
\[
\tilde{\rho}(\epsilon \eta) - \tilde{\rho}(\epsilon \xi) = \sum_{0 < |\beta| < N} (\xi - \eta)^\beta \partial^\beta \tilde{\rho}(\epsilon \xi) + r_N(\xi, \eta, \epsilon),
\]
(129)
where
\[
\begin{align*}
\sum_{|\beta| = N} |\beta| e^N (\xi - \eta)^\beta \int_0^1 (1 - t)^{|\beta| - 1} \partial^\beta \tilde{\rho}(\epsilon \eta + (1 - t)(\epsilon \xi - \epsilon \eta)) dt.
\end{align*}
\]
Substituting (126)-(128) into (124) and taking account of (129), we obtain
\[
(R_N(\epsilon) u)(x) = \sum_{\xi, \eta \in \mathbb{Z}^n} \tilde{a}(\xi - \eta) (2\pi i) \alpha \partial^\alpha \tilde{u}(\eta) e^{2\pi i \xi \cdot x}.
\]
The integral expression for the remainder \(r_N\) shows immediately that, for any \(k \geq 0\), there exists \(C > 0\) such that
\[
\begin{align*}
|r_N(\xi, \eta, \epsilon)| &\leq Ce^N |\xi - \eta|^N (1 + |\eta|)^{-k}, \quad \text{if } |\xi - \eta| \leq |\eta|/3, \\
|r_N(\xi, \eta, \epsilon)| &\leq Ce^N |\xi - \eta|^N, \quad \forall \xi, \eta,
\end{align*}
\]
(130)
where the first inequality comes from the fact that \(\tilde{\rho}\) belongs to the Schwartz space. In addition, since \(a = a(x) \in C^\infty(\mathbb{T}^n)\), we have the following decay property concerning the Fourier coefficients of \(a\) (see [7, Theorem 3.2.9]): for all \(r \geq 0\), there exists \(C > 0\) such that
\[
|\hat{a}(\xi - \eta)| \leq C(1 + |\xi - \eta|)^{-r}.
\]
(131)
To find the bound for the norm \(\|R_N(\epsilon) u\|_{H^{s+\sigma}(\mathbb{T}^n)}\), we specify \(v \in C^\infty(\mathbb{T}^n)\) and consider the dual product
\[
\langle R_N(\epsilon) u, v \rangle = \sum_{\xi, \eta \in \mathbb{Z}^n} \hat{a}(\xi - \eta) (2\pi i) \alpha \tilde{u}(\eta) \hat{v}(-\xi).
\]
(132)
We set
\[
U(\eta) = (1 + |\eta|^2)(s+m)/2|\tilde{u}(\eta)|, \quad V(\xi) = (1 + |\xi|^2)^{-(s+\sigma)/2}|\hat{v}(-\xi)|.
\]
Noting that \(|\alpha| = m\) and using Peetre’s inequality, there follows
\[
|(2\pi i)^\alpha \tilde{u}(\eta) \hat{v}(-\xi)| \leq C|\eta|^{|\alpha|} (1 + |\eta|^2)^{-(s+m)/2}(1 + |\xi|^2)^{(s+\delta)/2}U(\eta)V(\xi)
\]
\[
\leq C(1 + |\eta|^2)^{\delta/2}(1 + |\xi|^2)^{(s+\delta)/2}U(\eta)V(\xi)
\]
(133)
\[
\leq C(1 + |\eta|^2)^{\delta/2}(1 + |\xi|^2)^{s+\sigma/2}U(\eta)V(\xi)
\]
\[
\leq C(1 + |\eta|)^{\delta}(1 + |\xi|^2)^{s+\sigma/2}U(\eta)V(\xi),
\]
which, by taking account of (131), implies
\[ |\langle R_N(\epsilon)u, v \rangle| \leq C \sum_{\xi, \eta \in \mathbb{Z}^n} |r_N(\xi, \eta, \epsilon)|(1 + |\eta|)^\sigma(1 + |\xi - \eta|)^{s+\sigma-r}U(\eta)V(\xi). \] (134)

Noting that for $|\xi - \eta| \leq |\eta|/3$, we have, by virtue of (130),
\[ |r_N(\xi, \eta, \epsilon)| \leq C\epsilon^N|\xi - \eta|^N(1 + \epsilon|\eta|)^{-\sigma}, \]
and that for $|\xi - \eta| \geq |\eta|/3$, we have $(1 + |\eta|) \leq 3(1 + |\xi - \eta|)$, we thereby deduce that there exists $C > 0$ such that the summand of the right-hand side of (134) is bounded from above by
\[ C\epsilon^{N-\sigma}(1 + |\xi - \eta|)^{s+\sigma+N+\sigma-r}U(\eta)V(\xi), \]
and by
\[ C\epsilon^{N-\sigma}\left( \sum_{\xi, \eta \in \mathbb{Z}^n} (1 + |\xi - \eta|)^{s+\sigma+N+\sigma-r}U(\eta)^2 \right)^{1/2} \]
\[ \cdot \left( \sum_{\xi, \eta \in \mathbb{Z}^n} (1 + |\xi - \eta|)^{s+\sigma+N+\sigma-r}V(\xi)^2 \right)^{1/2}, \]
where we used the Cauchy-Schwarz inequality. Consequently, by taking $r$ sufficiently large, we arrive at
\[ |\langle R_N(\epsilon)u, v \rangle| \leq C\epsilon^{N-\sigma}\|U\|_{L^2(\mathbb{Z}^n)}\|V\|_{L^2(\mathbb{Z}^n)} \]
\[ \leq C\epsilon^{N-\sigma}\|u\|_{H^{s+m}(\mathbb{T}^n)}\|v\|_{H^{-s-\sigma}(\mathbb{T}^n)}, \] (135)
which proves (125).

\[ \square \]

**Proposition A.5.** Let $s \in \mathbb{R}$, $\rho$ and $\sigma_\epsilon$ defined by (117) and (119) with $\gamma > s + 1$, and let $a(x), |\alpha| = m$ be as in Lemma A.4. Then, there exists $C > 0$ such that
\[ \int_0^1 \|[a\partial^\alpha, \sigma_\epsilon u]\|_{L^2(\mathbb{T}^n)}^2 \epsilon^{-2(s+1)}(1 + \frac{\theta^2}{\epsilon^2})^{-1} \frac{d\epsilon}{\epsilon} \leq C\|u\|_{s+m-1,\Theta}^2, \] (136)
for all $u \in H^{s+m-1}(\mathbb{T}^n)$ and $0 < \Theta \leq 1$.

**Proof.** We replace $[a\partial^\alpha, \sigma_\epsilon u]$ by its expression (124) and we majorize each term. For $0 < |\beta| < N$, applying Proposition A.3 with $\gamma, s, \rho$ and $\sigma$, replaced by $\gamma - |\beta|$, $s - |\beta|$, $\rho^\beta$ and $\sigma_\epsilon^\beta$ yields
\[ \int_0^1 \|[\partial^\beta a\partial^\alpha u] \}^2_{L^2(\mathbb{T}^n)} \epsilon^{-2(s-|\beta|+1)}(1 + \frac{\theta^2}{\epsilon^2})^{-1} \frac{d\epsilon}{\epsilon} \leq C\|\partial^\beta a\partial^\alpha u\|_{s-|\beta|,\Theta}^2, \] (137)

which, noticing that $a \in C^\infty(\mathbb{T}^n)$, is in turn bounded by $C\|u\|_{s+m-1,\Theta}^2$ since $|\beta| \geq 1$. To majorize the remainder, we utilize (125) with $s$ replaced by $s-1$ and $\delta = 1-s$; we obtain
\[ \int_0^1 \|R_N(\epsilon)u\|_{L^2(\mathbb{T}^n)}^2 \epsilon^{-2s-2}(1 + \frac{\theta^2}{\epsilon^2})^{-1} \frac{d\epsilon}{\epsilon} \leq C\|u\|_{H^{s+m-1}(\mathbb{T}^n)}^2 \int_0^1 \epsilon^{2N-4}(1 + \frac{\theta^2}{\epsilon^2})^{-1} \frac{d\epsilon}{\epsilon}, \]
and taking $N \geq 3$, we have an upper bound given by $C\|u\|_{H^{s+m-1}(\mathbb{T}^n)}^2$.

\[ \square \]

**Remark 7.** Lemma A.4 and Proposition A.5 are still valid when the function $a = a(x)$ depends continuously on one parameter varying within a compact subset; in fact, it is sufficient to note that we can choose the constant $C$ in (131) independent of this parameter.
A.4. Sobolev spaces on the space $\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}$. The corresponding (more general) results in Propositions A.2-A.5 for the whole space $\mathbb{R}^n$ are available in [4], Chapter 2 and Chapter 4 (see also [2], Chapter 9, pp. 262 - 263). Here, we only state the corresponding results in Propositions A.2-A.5 for the product space $\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}$, the proof can be achieved by combining the proof above for the torus and the proof in [4] for the whole space.

We first fix some notations. Let $n_1, n_2$ be positive integers, and $x = (x_1, x_2) \in \mathbb{T}^{n_1} \times \mathbb{R}^{n_2}$. For $f = f(x) \in L^1(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$ and $\xi = (\xi_1, \xi_2) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, we define the Fourier coefficients at $\xi$ of $f$ by

$$\hat{f}(\xi) = \int_{\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}} f(x) e^{-2\pi i \xi \cdot x} \, dx.$$  \hspace{1cm} (138)

We consider the Schwartz space $\mathcal{S}(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$ of test functions, i.e. the functions which are infinitely differentiable, periodic in $x_1$ and rapidly decreasing in $x_2$ at infinity along with all partial derivatives. The tempered distributions space $\mathcal{S}'(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$ is then defined as the (continuous) dual of the Schwartz space $\mathcal{S}(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$.

For all $s \in \mathbb{R}$, the Hilbert spaces $H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$ are defined as follows:

$$H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}) = \{ f \in \mathcal{S}'(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}) \mid \| f \|_{H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})} < \infty \},$$

where the norm on $H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$ is defined as

$$\| f \|^2_{H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})} = \int_{\mathbb{T}^{n_2}} \sum_{\xi \in \mathbb{Z}^{n_1}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi_2,$$  \hspace{1cm} (139)

where $|\xi|^2 = |\xi_1|^2 + |\xi_2|^2 = \sum_{i=1}^{n_1} \xi_{1,i}^2 + \sum_{i=1}^{n_2} \xi_{2,i}^2$.

**Proposition A.6.** If $f \in H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$ and if there exists $C > 0$ such that

$$\| f \|^2_{H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})} \leq C,$$

for all $\theta \in (0, 1)$, then $f$ belongs to $H^{s+1}(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$.

We let $\rho = \rho(x_1, x_2) = \rho_1(x_1) \rho_2(x_2)$ be a smooth function on $\mathbb{R}^{n_1+n_2}$ satisfying the two properties in (118), and we define $\sigma_\epsilon$ on $\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}$ by setting $\sigma_\epsilon(x_1, x_2) = \sigma_{1\epsilon}(x_1) \rho_2(x_2)$ where

$$\sigma_{1\epsilon}(x_1) = \sum_{\xi \in \mathbb{Z}^{n_1}} \hat{\rho}_1(\epsilon \xi_1) e^{2\pi i \xi_1 \cdot x_1}.$$  \hspace{1cm} (140)

**Proposition A.7.** Let $s \in \mathbb{R}$, $\rho$ and $\sigma_\epsilon$ defined as above with $\gamma > s + 1$. Then there exists $C_1$ and $C_2 > 0$ such that

$$C_1 \| f \|^2_{H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})} \leq \| f \|^2_{H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})} + \| \sigma_\epsilon \|^2_{H^{s}(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})} \leq C_2\| f \|^2_{H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})},$$

for all $\theta \in (0, 1)$ and all $f \in H^s(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$.

**Proposition A.8.** Let $s \in \mathbb{R}$, $\rho$ and $\sigma_\epsilon$ defined as above with $\gamma > s + 1$, and let $\alpha$ be a multi-index with $|\alpha| = m$, and $a = a(x)$ belong to $C^\infty(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})$ and be constant outside a compact subset of $\mathbb{T}^{n_1} \times \mathbb{R}^{n_2}$. Then there exists $C > 0$ such that

$$\int_0^1 \| [a \partial^{\alpha}, \sigma_\epsilon u] \|^2_{L^2(\mathbb{T}^{n_1} \times \mathbb{R}^{n_2})} e^{-2(s+1)(1 + \frac{\theta^2}{\epsilon})^{-1}} \frac{d\theta}{\epsilon} \leq C \| u \|^2_{s+m-1, \theta},$$  \hspace{1cm} (141)
for all $u \in H^{s+m-1}(T^{n_1} \times R^{n_2})$ and $0 < \theta \leq 1$.

**Remark 8.** Proposition A.8 is still valid when the function $a(x)$ depends continuously on one parameter varying within a compact subset.

**Appendix B. Classical lemmas.** In this appendix, we collect some essential ingredients for the Sobolev spaces (see e.g. [23], Chapter 13 or [2], Appendix C).

**Lemma B.1.** Assume that $U$ is a regular open set of $R^d$, where $d$ is the dimension of the space.

i) Consider $u$ and $v$ which both belong to $L^\infty(U) \cap H^s(U)$ with $s > 0$. Then their product also belongs to $H^s(U)$ and there exists $C > 0$ depending only on $s$ and $U$ such that
\[
\|uv\|_{H^s(U)} \leq C(\|u\|_{L^\infty(U)}\|v\|_{H^s(U)} + \|v\|_{L^\infty(U)}\|u\|_{H^s(U)}).
\]

If $s > d/2$, then the $L^\infty$ assumption automatically follows from the Sobolev embedding, and we have the following estimate:
\[
\|uv\|_{H^s(U)} \leq C\|u\|_{H^s(U)}\|v\|_{H^s(U)}.
\]

ii) Let $F$ be a $C^\infty$ function on $R$ such that $F(0) = 0$. Then there exists a continuous function $C : [0,+\infty) \to [0,+\infty)$ such that for all $u \in H^s(U) \cap L^\infty(U)$ with $s \geq 0$:
\[
\|F(u)\|_{H^s(U)} \leq C(\|u\|_{L^\infty(U)})\|u\|_{H^s(U)}.
\]

If $s > d/2$, then the $L^\infty$ assumption automatically follows from the Sobolev embedding, and if furthermore we assume that $U$ is bounded and that $u$ is positive away from 0, i.e. $|u| \geq \epsilon_0$ for some $\epsilon_0 > 0$, then we have
\[
\frac{1}{u} \in H^s(U),
\]

if we choose $F$ to be a $C^\infty$ function such that $F(x) = 0$ for $|x| \leq \epsilon_0/2$ and $F'(x) = x/|x|$ for $|x| \geq \epsilon_0$.

iii) If $k$ is an integer greater than $d/2 + 1$ and $\alpha$ is a $d$-tuple of length $|\alpha| \in [1,k]$, there exists $C > 0$ depending only on $k$ and $U$ such that for all $a \in H^s(U)$ and all $u \in H^{\alpha - 1}(U)$, we have the following estimate:
\[
\|[\partial^\alpha, a]u\|_{L^2(U)} \leq C\|a\|_{H^s(U)}\|u\|_{H^{\alpha - 1}(U)}.
\]

iv) Let $\rho$ be a standard mollifier, i.e. $\rho \in D(R^d)$ and $\int_{R^d} \rho = 1$, and set $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$, then for all $a \in W^{1,\infty}$ and $u \in L^2$, for $j \in \{1, \cdots, d\}$, we have
\[
\|[a\partial_j, \rho_\epsilon u]\|_{L^2} = \|[a\partial_j(\rho_\epsilon * u) - \rho_\epsilon * (a\partial_j u)]\|_{L^2} \leq C\|a\|_{W^{1,\infty}}\|u\|_{L^2},
\]

where the constant $C$ depends on $d$ and $\rho$, but is independently of $a$, $u$ and $\epsilon$.

For the weighted Sobolev spaces $H^s_{\gamma}(U)$ defined in Subsection 2.5, we also have the following Lemma.

**Lemma B.2.**

i) For all $q, r, s$ with $r + s > 0$ and $q \leq \min(r, s)$, $q < r + s - d/2$, there exists $C > 0$ such that for $a \in H^r$ and all $u \in H^s_{\gamma}$,
\[
\|au\|_{H^s_{\gamma}} \leq C\|a\|_{H^r}\|u\|_{H^s_{\gamma}}.
\]
ii) If $m$ is an integer greater than $d/2+1$ and $\alpha$ is a $d$-tuple of length $|\alpha| \in [1, m]$, there exists $C > 0$ depending on $d$ and $\alpha$ such that for all $\gamma \geq 1$, for all $u$ in $H^m$ and all $v \in H^{(1)}_{\alpha-1}$, 
\[
\|e^{-\gamma t}[\partial^\alpha, a]u\|_{L^2} \leq C\|a\|_{H^m} \|u\|_{H^{(1)}_{\alpha-1}}.
\]

iii) If $m$ is an integer greater than $d/2+1$ and $k$ is a positive integer less than $m$, and $\alpha$ is a $d$-tuple of length $|\alpha| \in [1, m-k]$, there exists $C > 0$ depending on $d, m$ and $k$ such that for all $\gamma \geq 1$, for all $u$ in $H^m$ and all $v \in H^{(1)}_{\alpha+k-1}$, 
\[
\|[\partial^\alpha, a]u\|_{H^{k}} \leq C\|a\|_{H^m} \|u\|_{H^{(1)}_{\alpha+k-1}}.
\]

The first two estimates are stated in [2, Chapter 9] and the last one is obtained by repeatedly using the second estimate.

Appendix C. Trace theorem. In this Appendix we give a trace theorem which we used in the article. See Appendix A in [22] for some other trace results.

We fix $p$ such that $1 < p < +\infty$, and we also let $q = p'$ the conjugate exponent of $p$ (i.e. $1/p + 1/q = 1$).

**Lemma C.1.** Let $X$ be a reflexive Banach space, and let $\lambda = \lambda(x) \in C[0,1]$ and $\lambda \geq c_0$ for some positive constant $c_0$. Assume that two sequences of function $u^\epsilon, g^\epsilon \in L^p_x(0,1;X)$ satisfy
\[
\begin{aligned}
-\epsilon u^\epsilon_{xx} + \lambda(x)u^\epsilon = g^\epsilon, \\
u^\epsilon(0) = u_0, \\
u^\epsilon(1) = 0,
\end{aligned}
\]  
with $g^\epsilon$ bounded in $L^p_x(0,1;X)$ independent of $\epsilon$ and $u_0 \in X$. Then the sequence $u^\epsilon$ is bounded in $L^p_x(0,1;X)$ independently of $\epsilon$, and for any subsequence $u^\epsilon \to u, g^\epsilon \to g$ (strongly or weakly) converging in $L^p_x(0,1;X)$, we have that $u^\epsilon(0)$ converges to $u(0)$ in $X$ (weakly at least), and hence $u(0) = u_0$.

**Proof.** For simplicity, we drop $\epsilon$, and by solving (142), we obtain that
\[
u^\epsilon(x) = \int_x^1 \frac{1}{\epsilon} e^{-f_{x}^{\epsilon \lambda(x)/\epsilon d\tilde{x}}} g(x_1) dx_1. \tag{143}
\]
Taking the $X$-norm of (143) and raising it to the $p$-th power, and using Hölder’s inequality, we find
\[
\|\nu^\epsilon\|_X^p \leq \left( \int_x^1 \frac{1}{\epsilon} e^{-f_{x}^{\epsilon \lambda(x)/\epsilon d\tilde{x}}} dx_1 \right)^{p/q} \int_x^1 \frac{1}{\epsilon} e^{-f_{x}^{\epsilon \lambda(x)/\epsilon d\tilde{x}}} \|g(x_1)\|_X^p dx_1. \tag{144}
\]
Direct computations show that the first integral in (144) is less than
\[
\int_x^1 \frac{1}{\epsilon} e^{-f_{x}^{\epsilon \lambda(x)/\epsilon d\tilde{x}}} dx_1 = \int_x^1 \frac{1}{\epsilon} e^{-c_0(x_1-x)\epsilon} dx_1 \leq \frac{1}{c_0},
\]
then integrating (144) from 0 to 1 with respect to $x$ and switching the order of integration for $x, x_1$, we finally arrive at
\[
\int_0^1 \|\nu^\epsilon\|_X^p dx \leq \left( \frac{1}{c_0} \right)^p \int_0^1 \|g\|_X^p dx. \tag{145}
\]
Therefore, it follows that the $u^\epsilon$ are uniformly bounded in $L^p_x(0,1;X)$. The existence of the trace $u(0)$ and its linear continuous dependence on $\{u, g\}$ follow easily. \qed
Remark 9. If two sequences $u^\epsilon, g^\epsilon \in L^p_x(0, 1; X)$ in Lemma C.1 satisfy, instead of (142):

$$
\begin{align*}
-\epsilon u^\epsilon_{xx} - \lambda(x)u^\epsilon_x &= g^\epsilon, \\
 u^\epsilon(1) &= u_1, \quad u^\epsilon_x(0) = 0,
\end{align*}
$$

then we still have that the $u^\epsilon_x$ is bounded in $L^p_x(0, 1; X)$ independently of $\epsilon$, and also for any subsequence $u^\epsilon \to u, g^\epsilon \to g$ (strongly or weakly) converging in $L^p_x(0, 1; X)$, $u^\epsilon(1)$ converges to $u(1)$ in $X$ (weakly at least), and hence $u(1) = u_1$.

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