The equations of moist advection: a unilateral problem

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In an earlier article, the authors discussed the uniqueness of solutions for moist advection problems. This article dealt with a simplified case in which the saturation specific humidity \(q_s\) – which is a slightly varying function of temperature – was assumed to be constant. In trying to extend the results of Temam and Tribbia to the case where \(q_s\) is not constant, it appeared that the equations of moist advection in this case were not coherent in the extreme cases where the atmosphere is totally dry or totally humid.

The aim of this article is to describe this difficulty and propose a mathematically satisfying solution in the context of so-called variational inequalities.

Key Words: advection equations; variational inequalities; humid atmosphere with saturation; discontinuous equations

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1. Introduction

It is known that our lack of understanding of the physics of clouds is a major cause of uncertainty in current weather prediction models. The numerous related difficulties include the number of scales involved for clouds: from outer dimensions of 100–1000 km to sizes of the order of a micron of the particles and droplets of water from which they are composed. Another major difficulty that we started to address in Temam and Tribbia (2013) and in related more mathematical references (Coti Zelati and Temam, 2012; Coti Zelati et al., 2013) is due to the changes of phase occurring in the cloud through e.g. evaporation and precipitation. These changes of phase are mathematically expressed by discontinuities, which render difficult the theoretical understanding of the corresponding equations, which are hence nonlinear and discontinuous. In the case where the saturation specific humidity \(q_s\) is assumed to be constant, these equations have been studied in earlier articles: in Coti Zelati and Temam (2012), mathematical modelling and the issue of the existence and regularity of solutions were addressed for the \(T - q\) system (where \(T\) denotes the temperature and \(q\) the content of water vapour in the air) in the case where the velocity \(u\) of the air was prescribed. Coti Zelati et al. (2013) addressed the additional issue of uniqueness for the same system, a result also presented in Temam and Tribbia (2013) for the physics-oriented reader, in the spatially homogeneous case. Finally, in Coti Zelati et al. (2015) we addressed the even more mathematically involved case when \(u\) is not prescribed, leading to a nonlinear discontinuous system coupling \(T\), \(q\) and \(u\).

As we said, all these results and articles relate to the case where \(q_s\) is assumed to be constant. In trying to extend some of these results to the case where \(q_s\) is not constant, we encountered the difficulty that the equations for moist advection are not coherent in this context in extreme situations where the atmosphere is totally dry \((q = 0)\) or totally humid \((q = 1)\). In section 2, we recall the equations of moist advection concerning \(T\), \(q\) and \(q_s\), borrowed from classical references, and we explain the lack of coherence that they contain. Then in section 3 we propose a mathematically satisfying solution of this difficulty in the context of the so-called variational inequalities, which were introduced in the engineering and mathematical literature to handle unilateral problems; see e.g. Lions (1969), Brézis (1972), Duvaut and Lions (1976), Kinderlehrer and Stampacchia (1980), Frémond (2002, 2012).

2. Equations of moist advection

The system that we consider is the basic system for moist advection, described in e.g. Haltiner (1971), Haltiner and Williams (1980) and Rogers and Yau (1989). The equations for the temperature \(T\) and the content \(q\) of water vapour in the air satisfy the equations

\[
\frac{\partial T}{\partial t} + v \cdot \nabla T + c_p \frac{\partial q}{\partial p} + \frac{\partial T}{\partial p} + \frac{\partial}{\partial p} (\alpha T) = - \frac{\mu T}{\rho}, \quad (1)
\]

\[
\frac{\partial q}{\partial t} + v \cdot \nabla q + \omega = \frac{S}{\rho}. \quad (2)
\]

Here \(u = (v, \omega)\) is the (given) velocity of the fluid in the \(x, y, p\) coordinate system; \(\nabla = (\partial_x, \partial_y)\) is the horizontal gradient; \(c_p\) is the specific heat capacity of air at constant pressure; \(S\) represents any sources or sinks of water vapour in mass per unit time and \(\rho\) is the density (see below). Equations (1) and (2) include the dissipation operators \(\mathcal{A}_T, \mathcal{A}_q\), which do not appear in the references quoted above:

\[
\mathcal{A}_T = -\mu T \Delta T - \nu_T \frac{\partial}{\partial p} \left( \frac{g p}{RT} \right)^2 \frac{T}{\partial p}, \quad (3)
\]
\( A \frac{dq}{dp} = -\mu \frac{\partial q}{\partial x} - v_0 \frac{\partial}{\partial t} \left[ \left( \frac{g \rho}{RT} \right)^2 \frac{\partial T}{\partial p} \right] \) \tag{4}

Here \( \mu, v_0, \mu, v \) are turbulent positive viscosity coefficients and \( \Delta = \frac{1}{\nu} + \frac{1}{\nu} \) is the horizontal Laplacian; \( \overline{T} = T(p) \) is the average temperature over the isobaric surface with pressure \( p \). Finally,

\[ F = F(T) = q_0 T \left( \frac{LR - c_p R_0 T}{c_p R_0 T^2 + q_0 L^2} \right), \tag{5} \]

\[ \delta = 1 \text{ for } \omega < 0 \text{ and } q > q_i, \tag{6} \]

\[ \delta = 0 \text{ for } \omega \geq 0 \text{ or } q < q_i. \tag{7} \]

Here \( R \) is the gas constant for air, \( R_c \) is the gas constant for water vapour and \( L \) is the latent heat of condensation of water; this is a slightly varying function of \( T \) that we assume constant for simplicity; see e.g. Gill (1982) and Pedlosky (1987). Finally \( q_i \) is the saturation specific humidity, which reduces to \( d \) in the spatially homogeneous case. As we mentioned earlier, we studied these equations in Coti Zelati and Temam (2012), Coti Zelati et al. (2013) and Temam and Tribbia (2013) in the case where \( q_i \) was assumed to be constant.

In trying to extend these results to the more general case where \( q_i \) satisfies Eqs (8) and (9), we found that Eq. (2) cannot be satisfied in the extreme cases where \( q = 0 \) ( totally dry atmosphere) and \( S < 0 \) or \( q = 1 \) (atmosphere totally humid) and \( S > 0 \). In the first case, for instance, if \( q = 0 \) then the atmosphere is totally dry and the presence of a sink \( S < 0 \) would tend to decrease \( q \), which is physically impossible. Similarly, when \( q = 1 \) the atmosphere is totally humid and the presence of a source \( S > 0 \) will tend to increase \( q \), which is physically impossible. We explain in section 3 how we propose to modify Eq. (2) to solve this difficulty.

3. Mathematical reformulation

As in Temam and Tribbia (2013), here we will restrict ourselves to the spatially homogeneous case, leaving the general case, which involves more heavy mathematical tools, to the article Temam and Wu (2015). However, as in Temam and Tribbia (2013), the guiding ideas are the same in both situations.

The equation that needs to be modified is Eq. (2); we use \( \mathcal{L}(q) \) to denote the left-hand side of Eq. (2), that is

\[ \mathcal{L}(q) = \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q + \omega q + A q, \]

which reduces to \( dq/dt \) in the spatially homogeneous case. As we said, when \( q = 0 \), Eq. (2) reduces to

\[ \frac{dq}{dt} = \frac{S}{\rho}, \tag{10} \]

and this is impossible if \( S < 0 \), because \( q \) cannot decrease further. Physically speaking, for \( q = 0 \), \( dq/dt \geq 0 \), so that instead of Eq. (10) we have

\[ \mathcal{L}(q) \geq \frac{S}{\rho}. \tag{11} \]

Similarly, for \( q = 1 \), \( dq/dt \leq 0 \) and Eq. (10) cannot be true if \( S > 0 \). In this case we have

\[ \mathcal{L}(q) \leq \frac{S}{\rho}. \tag{12} \]

In summary, we should have

\[ \mathcal{L}(q) \begin{cases} \frac{\rho}{S} & \text{for } q = 0, \\ \frac{\rho}{S} & \text{for } 0 < q < 1, \\ \frac{\rho}{S} & \text{for } q = 1. \end{cases} \tag{13} \]

Now, let \( q^b \) be any number (or function of \( t \)) between 0 and 1, like \( q \). We easily see that Eq. (13) implies

\[ \mathcal{L}(q) - \frac{S}{\rho} (q^b - q) \geq 0. \tag{14} \]

Furthermore, requiring Eq. (14) for any \( 0 \leq q^b \leq 1 \) is equivalent to requiring Eq. (13).

Equations like (14) have been broadly discussed in various areas of engineering, mathematics and physics in the context of unilateral phenomena or phenomena with thresholds: semi-permeable media, temperature control, heat control, problems with friction in elasticity and viscoelasticity, plasticity, etc.; see e.g. Duvaut and Lions (1976), Frémond (2002, 2012) and references therein. The formulation of problems similar to Eq. (14) was called a variational inequality and such problems were studied in e.g. Brézis (1972), Duvaut and Lions (1976), Kinderlehrer and Stampacchia (1980) and Ekeland and Temam (1999).

In this context, we can rephrase Eq. (14) as follows:

\[ \text{to find a function } q \text{ from } (0, t_1) \]

\[ \text{into } K = [0, 1], \text{ such that} \]

\[ \left( \mathcal{L}(q) - \frac{S}{\rho} \right) (q^b - q) \geq 0, \tag{15} \]

for every function \( q^b \) from \( (0, t_1) \) into \( K \).

We can now couple this inequality with Eq. (1), which remains an equality. We are looking for a function \( T \) from \( (0, t_1) \) into \( \mathbb{R} \) and a function \( q \) from \( (0, t_1) \) into \( [0, 1] \), such that

\[ c_p \frac{dT}{dt} - RT \omega = -\delta F \omega, \tag{16} \]

\[ \left( \mathcal{L}(q) - \frac{\delta}{\rho} \omega \right) (q^b - q) \geq 0, \tag{17} \]

for all functions \( q^b \) on \([0, t_1]\) with values in \([0, 1] \).

We then need to address the fact that \( \delta \) is discontinuous. We proceed here exactly as in Temam and Tribbia (2013), by introducing the multivalued Heaviside function \( H \), which is equal to 0 for \( x < 0 \), 1 for \( x > 0 \) and \([0, 1] \) for \( x = 0 \). Then, in view of Eqs (16) and (17) and what was done in Temam and Tribbia (2013), we are now looking for a real function \( T \) on \((0, t_1)\) and for functions \( q, h \) and \( h_T \) from \((0, t_1)\) into \([0, 1] \), such that

\[ c_p \frac{dT}{dt} - RT \omega = -h_T f(T, q), \tag{18} \]

\[ \left( \mathcal{L}(q) - h_T f(T, q) \right) (q^b - q) \geq 0, \tag{19} \]

for all functions \( q^b \) from \((0, t_1)\) into \([0, 1] \).

Here, as in Temam and Tribbia (2013), we have used the notation

\[ f(T, q) = -\frac{F \omega -}{p}, \tag{20} \]

\[ \omega = \max(-\omega_1, 0). \]

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3.1. Remark 1

Note that the number one plays totally different roles for $0 \leq q \leq 1$ and $0 \leq h_T$, $h_q \leq 1$: $0 \leq q \leq 1$ results from the physical definition of $q$, whereas $0 \leq h_T$, $h_q \leq 1$ results from the fact that $h_T$, $h_q$ are representative of the multivalued function $H(q - q_0)$ for the $T$ equation and the $q$ equation, respectively.

3.2. Remark 2

In Eq. (19), $L(q) = dq/dt$. We temporarily keep the notation $L(q)$ as a reminder that $L(q)$ is the left-hand side of Eq. (2) in the non-homogeneous case.

Finally, Eqs (18) and (19) have to be supplemented with Eq. (9) for $q_0$, which becomes

$$\frac{dq_0}{dt} = h_f(T, q),$$

and we need to specify the initial conditions for $T, q$ and $q_0$:

$$T(0) = T_0, \quad q(0) = q_0, \quad q_0(0) = q_{00}. \quad (22)$$

We will show briefly in section 4 how one can prove the existence of solutions for the problem (Eqs (18)–(22)).

4. Existence of solutions

The functions $F$ and $f$ are singular in the physically irrelevant case where $T$ and $q_0 = 0$. Hence we replace them by $F^+$ and $f^+$:

$$F^+_r = F^+_r(T, q_0) = q_0^+ \left[ \frac{(L - C_p R_T)T}{C_p R_T \max(T, r)^2 + q_0^+ L^2} \right],$$

$$f^+_r = f^+_r(T, q) = -\frac{p \omega}{p},$$

with $r$ a positive number smaller than any temperature on Earth. For mathematical convenience and to ensure the positivity of $F$ and $f$, we have replaced $(L - C_p R_T)T$ by its positive part $(L - C_p R_T)^+$, observing that $T \leq L/R_c R_T$ for any temperature on Earth. We have also replaced $q_0$ by $q_0^+$; this is done temporarily until we prove that $q_0 \geq 0$.

The discontinuity of $H$ is handled as in Coti Zelati and Temam (2012) (and subsequent works), replacing it by its regularized form:

$$H_{\varepsilon_1}(s) = \begin{cases} 0 & \text{for } s \leq 0, \\ \frac{s}{\varepsilon_1} & \text{for } 0 \leq s \leq \varepsilon_1, \\ 1 & \text{for } s \geq 1, \end{cases}$$

where $\varepsilon_1 > 0$ is a small parameter.

Then, the fact that $0 \leq q \leq 1$ is approximately enforced by the so-called penalization method (see Courant, 1943; Lions, 1969; Polak, 1971). Hence we now replace Eqs (18), (19), (21) and (22) by

$$\frac{dT^e}{dt} = -\frac{RT^e \omega}{p} - H_{\varepsilon_1}(q - q_0^+)f^+_r(T^e, q^e),$$

$$\frac{dq^e}{dt} = -\frac{1}{\varepsilon_2}(q^e)^- + \frac{1}{\varepsilon_2}(q^e - 1)^- + H_{\varepsilon_1}(q^e - q_0^+)f^+_r(T^e, q^e),$$

and

$$T^e(0) = T_0, \quad q^e(0) = q_0, \quad q_0^e(0) = q_{00}^e. \quad (26)$$

Here, $\varepsilon_2$ is another small $> 0$ parameter, namely the penalty parameter, and $x^- = \max(-x, 0), x^+ = \max(x, 0)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$.

The existence and uniqueness of solutions for Eq. (22)–(25) is elementary. Now we show how to derive a priori estimates which are independent of $\varepsilon$ for $T^e$, $q^e_0$ and their time derivatives.

4.1. A priori estimates

We show here how to derive a priori estimates for $T^e$, $q^e_0$ and $q^e_0$, which are then used to pass to the limit $\varepsilon = (\varepsilon_1, \varepsilon_2) \to 0$.

We first observe that Eq. (25) and $q_{00}^e = 0$ in Eq. (26) imply that $q^e_0 \geq 0$ for all time, so that $(q^e_0)^+ = q^e_0$. We assume that the vertical coordinate $p$ varies between $p_0$ and $p_1$, $0 < p_0 \leq p \leq p_1$ and that $\omega$ is bounded. Hence $f^e_r$ and $f^e_0$ are uniformly bounded for all $T$ and $q_0$.

The first a priori estimates are obtained by multiplying Eq. (23) by $T^e$ and Eq. (24) by $q^e$ and integrating in time; we find

$$\frac{c_p}{2} \frac{dT^e}{dt} \leq \frac{R_0}{p} |T^e|^2 - H_{\varepsilon_1}(q^e - q_0^+)f^+_r(T^e, q^e),$$

$$\frac{d|T^e|^2}{dt} \leq c(|T^e|^2 + 1), \quad (27)$$

$$\frac{1}{\varepsilon_2} \frac{d(q^e)^2}{dt} - \frac{1}{\varepsilon_2}(q^e)^- + \frac{1}{\varepsilon_2}(q^e - 1)^- + H_{\varepsilon_1}(q^e - q_0^+)f^+_r(T^e, q^e)$$

$$\leq c(q^e)^- \leq c(||q^e||^2 + 1). \quad (28)$$

Here the $c$ denote suitable constants independent of $\varepsilon$, which may change from one place to another. We then observe that

$$\frac{1}{\varepsilon_2} \frac{d(q^e)^-}{dt} + \frac{1}{\varepsilon_2}(q^e - 1)^- + q^e$$

$$= \frac{1}{\varepsilon_2}(q^e)^2 - \frac{1}{\varepsilon_2}(q^e - 1)^2 + \frac{1}{\varepsilon_2}(q^e - 1)^2 + \frac{1}{\varepsilon_2}(q^e)^- \geq 0. \quad (29)$$

Hence we infer from Eqs (27) and (28), using the so-called Gronwall lemma, that $T^e$ and $q^e$ are uniformly bounded on $(0, t_1)$ independently of $\varepsilon$:

$$\sup_{0 \leq t \leq t_1} (|T^e(t)| + |q^e(t)|) \leq c. \quad (30)$$

In addition, integrating each side of Eq. (28) from 0 to $t_1$, we obtain, in view of Eq. (29), that

$$\int_0^{t_1} \left[ (q^e)^2 + (q^e - 1)^2 \right] dt \leq c\varepsilon_2. \quad (31)$$

Returning then to Eq. (23), we observe that $dT^e/dt$ is also uniformly bounded, from which we deduce that the functions $T^e$ are equicontinuous and thus the sequence contains a subsequence which converges uniformly.

To obtain a similar result for $q^e$, we multiply Eq. (24) by $-(q^e)^- / \varepsilon_2$, which yields

$$\frac{1}{2\varepsilon_2} \frac{d}{dt} \left[ (q^e)^2 + \frac{1}{\varepsilon_2}(q^e)^2 + \frac{1}{\varepsilon_2}(q^e - 1)^2 \right]$$

$$= H_{\varepsilon_1}(q^e - q_0^+)f^+_r(T^e, q^e)$$

$$\leq \varepsilon_2 \frac{(q^e)^-}{\varepsilon_2} \leq \frac{1}{2} (q^e)^2 + c. \quad (32)$$

Observing that $(q^e - 1)^2 (q^e)^- = 0$, we deduce from Eqs (31) and (32) (after integration from 0 to $t_1$) that $d\tilde{q}^e / dt$ is bounded in $L^2(0, t_1)$ and the sequence $\tilde{q}^e$ is thus uniformly equicontinuous on $(0, t_1)$. With an additional extraction of subsequence, we find that $\tilde{q}^e$ converges uniformly on $(0, t_1)$ to some limit $q$ and Eq. (31) guarantees that $0 \leq q \leq 1$.

The passage to the limit in Eqs (23)–(26) is then straightforward and we obtain the existence of a solution for Eqs (18)–(22) (with $F$ and $f$ replaced by $F_r$ and $f_r$). The details for the non-homogeneous case just appeared in Temam and Wu (2015).
4.2. Remark 3

Although we are able to show that the time derivatives remain bounded in some sense, we do not expect higher derivatives to remain bounded, because the solutions of unilateral problems are not (highly) smooth in general.

4.3. Remark 4 (concluding remarks)

We do not speculate in this article on the possible utilization of this formulation for actual numerical simulations, but we thought it useful to bring this difficulty to the attention of the specialists. We observe also that, in the spatially non-homogeneous case, the atmosphere can become totally dry or totally wet in some regions, at least at the precision of the numerical simulations, and one may elaborate on the practical importance of this difficulty and on the significance of numerical simulations solving an incorrect problem. It could also be that some automatic procedures inserted in the codes to enforce the constraint \(0 \leq q \leq 1\) amount to some kind of approximation of the proposed variational inequality.

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