RECENT PROGRESSES IN BOUNDARY LAYER THEORY

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Abstract. In this article, we review recent progresses in boundary layer analysis of some singular perturbation problems. Using the techniques of differential geometry, an asymptotic expansion of reaction-diffusion or heat equations in a domain with curved boundary is constructed and validated in some suitable functional spaces. In addition, we investigate the effect of curvature as well as that of an ill-prepared initial data. Concerning convection-diffusion equations, the asymptotic behavior of their solutions is difficult and delicate to analyze because it largely depends on the characteristics of the corresponding limit problems, which are first order hyperbolic differential equations. Thus, the boundary layer analysis is performed on relatively simpler domains, typically intervals, rectangles, or circles. We consider also the interior transition layers at the turning point characteristics in an interval domain and classical (ordinary), characteristic (parabolic) and corner (elliptic) boundary layers in a rectangular domain using the technique of correctors and the tools of functional analysis. The validity of our asymptotic expansions is also established in suitable spaces.

1. Introduction. We review in this article some recent developments in the study of singularly perturbed problems, that is, (initial and) boundary value problems that contain a small parameter affecting the highest order derivatives. The engineering literature on boundary layers is very vast and includes the theory of Prandtl [114, 115] in fluid mechanics as well as works by von Kármán [141], [121], and others, and the beautiful experimental book of Van Dyke [27]. On the theoretical side, the study of singular perturbations and boundary layers remains very challenging, and includes the notorious problem of turbulent boundary layer [121]. Nevertheless such problems have attracted the attention of mathematicians during the last 60 years or so, and some substantial insight has been gained in such problems; see, e.g.,

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One classical motivation of studying singularly perturbed problems is from the so-called vanishing viscosity limit in fluid mechanics, see, e.g., [10, 15, 24, 28, 80, 82, 84, 86, 96, 100, 109, 111, 112, 125, 131]. Beside the vanishing viscosity problem which underlies this article, one can mention many singular perturbation problems, in particular in geophysical fluid mechanics, in relation with rotating fluids and Ekman layers, see, e.g., [16, 46]. Returning to the vanishing viscosity limit, we recall that the motion of viscous and inviscid fluids is modeled respectively by the Navier-Stokes and Euler equations. Considering the Navier-Stokes equations at small viscosity as a singular perturbation of the Euler equations, a major problem, still essentially open, is the asymptotic behavior of the Navier-Stokes solutions, i.e., to verify if the Navier-Stokes solutions converge, in some function spaces, to the Euler solutions as the viscosity tends to zero. This is an outstanding fundamental problem in analysis. Because an inviscid flow of the Euler type is free to slip along the boundary while any viscous flow of the Navier-Stokes type must adhere to the boundary, boundary layers must occur at small viscosity. Of course controlling the boundary layers of singularly perturbed problems is the main key to understanding the nature of vanishing viscosity limit.

Another important motivation for singular perturbations is associated with the numerical computation of solutions to those problems. In approximating solutions to singularly perturbed boundary value problems, it is well-known that a very large (discrete) gradient is created near the boundary. Hence, in most classical simulations of singularly perturbed equations, a drastic mesh refinement is usually required near the boundary to obtain an accurate approximation of the solution; a fine mesh of order $\varepsilon^{1/2}$, where $\varepsilon$ is the non-dimensional viscosity, is usually suggested [126]. Departing from massive mesh refinements, some new semi-analytic methods have been proposed and successfully applied under the names of enriched spaces, extended finite element method (XFEM), and generalized finite element method (GFEM). The common idea is to add to the Galerkin basis or its analogue in finite differences, finite elements, or finite volumes, some specific shape functions which carry the inherent singularity of the problem. For singular perturbations, the specific shape functions are related to the boundary layer correctors computed analytically; in crack theory, the shape functions embolden the singularity at the tip of a crack. See [3, 4, 106] for the XFEM method for cracks; see [104, 105] for GFEM; and see e.g. [18, 19, 122] for singular perturbations as well as the articles [65, 66, 67, 71, 74, 76] which this article covers in part and generalizes. Such methods have proven to be highly efficient without any need of mesh refinement near the boundary.

In this review article, we consider several classes of singular perturbation problems in some non-classical settings. Summarizing results in the recent works [35, 36, 38, 39, 40, 41, 43, 44, 72, 134], we study, in Section 2, the boundary layers in a smooth domain with curved boundary; see also [56, 57, 69, 132, 133] for the case of a flat boundary. Toward this end, we first recall some elements of differential geometry and give some concrete examples of the domains under consideration in Section 2.1. In Sections 2.2 and 2.3, boundary layers of the reaction diffusion and heat equations are respectively investigated. Here we construct an asymptotic expansion of the singularly perturbed reaction-diffusion solution or the heat solution at an arbitrary order with respect to the small diffusivity. The point of view that
we systematically use in this article (and others) is the utilization of correctors as proposed by J. L. Lions [91]. The corrector is in fact the solution of an equivalent of the Prandtl equation [114, 115, 129] for the problem and it accounts for the rapid variation of the functions and their normal derivatives in the boundary layer. It also corrects (hence the name) the discrepancies between the boundary values of the viscous and inviscid solutions. This construction is, of course, closely related to the matching asymptotic method, see, e.g., [29, 30, 88]. Hence the analysis is at first informal and of a physical nature. Then the rigorous validity of our asymptotic expansions is confirmed globally in the whole domain by performing energy estimates on the difference of the diffusive solution and the proposed expansion. Our expansion at an arbitrary order with respect to the small diffusivity provides the complete structural information of the boundary layers.

In Section 3, we discuss a class of singularly perturbed problems with a turning point in an interval domain. The literature about the analysis of singularly perturbed problems with a turning point is not very large; see however [7, 22, 23, 68, 108, 127, 144, 145, 146, 147, 148]. Here we follow [73, 79]. The cases where the limit problem is compatible and non-compatible with the given data are considered. With limited compatibility conditions on the data, the asymptotic expansions can be constructed only up to the order allowed by the level of compatibilities. However, using a smooth cut-off function compactly supported around the turning point, which localizes the singularities due to the non-compatible data, we obtain the asymptotic expansions up to any order.

In Section 4, we consider domains with corners. Corners generate singularities which have to be corrected by boundary layers techniques, and/or they affect existing boundary layers due to the singularly perturbed nature of the equations.

Of the first type, and without any “small coefficient” in the equation, are the singularities created at \( t = 0 \) by the incompatible initial and boundary data (see [124, 128, 17, 32, 33]) or the singularities created by corners in a geometric domain (see [51, 52] and [8, 58, 149], and the references therein). All these singularities develop already with regular differential operators, that is, differential operators with order one coefficients for the leading derivatives. Another type of problems on which we will focus, concerns the interaction of corner singularities with the boundary layers due to singularly perturbed differential operators, that is, differential operators with a small coefficient affecting the highest derivatives. The remarkable article [122] illustrates the variety and complexity of the boundary layers that occur for a singularly perturbed elliptic differential operator of the second order in a square. These problems have been studied in a variety of contexts in, e.g., [94, 95] and in, e.g., the articles [36, 42, 44, 75] on which this section is partly based. In this section, we study the asymptotic behavior of solutions to a convection-diffusion equation in a rectangular domain \( \Omega \). The Dirichlet boundary condition is then supplemented along the edges and at the corners. The elliptic corner layers are introduced to handle the interaction of the parabolic and classical boundary layers at the four corners. Our analysis simplifies that of [122] by minimizing the construction of boundary layers needed and extends the asymptotic expansions of [122] up to any order.

In summary aiming to study the asymptotic behavior of the solutions to some singular perturbation problems, we construct the boundary layer (or interior layer) correctors and obtain the full structural information of
(a) the boundary layers of the reaction-diffusion equation in a smooth curved domain
(b) the boundary and initial layers of the heat equation in a smooth curved domain
(c) the interior layers of the convection-diffusion equation with turning points in an interval domain
(d) the interaction of the boundary and corner layers for the convection-diffusion equation in a rectangular domain

This article is not meant to be exhaustive in any way. Besides the topics of this article, more subjects will be covered in a forthcoming book [37]. Not covered in this article is the case of convection-diffusion equations where the limit problem is hyperbolic. This project was studied from various point of views in, e.g., [5, 45, 85, 89, 136, 138]. Our contributions on the subject appear in [77, 78], in the review article (on the subject) [70], and in [66] for some computational aspects.

Other problems not discussed in this article include many problems of classical and geophysical fluid flows with small viscosity; see, e.g., [16, 24, 27, 34, 36, 38, 40, 41, 43, 44, 49, 50, 54, 55, 56, 57, 59, 80, 81, 83, 93, 94, 95, 96, 100, 102, 103, 107, 116, 123, 130, 132, 133, 134, 135]; electromagnetism, acoustic, and the Helmholtz equation [2, 9, 117, 118]; see also in e.g. [53] and in the references therein the issue of boundary layers for hyperbolic equations; the issue already mentioned of the numerical approximation of singularly perturbed problems, in particular in the context of enriched spaces [18, 19, 60, 61, 62, 64, 65, 66, 67, 71, 74, 76, 85, 104, 105, 106, 113, 119, 126, 127]; the important subject of vanishing viscosity for the Hamilton-Jacobi equations which is a whole subject by itself; see among many references, [21, 92]. See also various perspectives in singular perturbations and boundary layers in [1, 12, 13, 14, 25, 26, 31, 53, 47, 48, 90, 99, 101, 120, 137, 142, 143]. Let us mention also the article [97] and the subsequent book [98] which offer totally new perspectives in boundary layer separation. Finally let us mention that we considered convection-diffusion equations in a circular domain where two characteristic points appear. The singular behaviors may occur at these points depending on the behavior of the given data. However, this case is not covered here and the reader is referred to the review article [70]. Convection-diffusion equations in general curved domains will be studied elsewhere.

2. Boundary layers in a curved domain in $\mathbb{R}^n$, $n = 2, 3$.

2.1. Elements of differential geometry.

The domain $\Omega$ is assumed to be bounded and smooth in $\mathbb{R}^3$. In this section, we construct an orthogonal curvilinear coordinate system adapted to the boundary $\Gamma := \partial \Omega$ and write some differential operators with respect to the curvilinear system. Any smooth and bounded domain in $\mathbb{R}^2$ can be handled in a similar (but easier) manner by suppressing the second tangential variable as in Section 2.1.2 below. More information about the geometry and construction of a special coordinate system is well-described in, e.g., [6, 20, 87] as well as [40, 43].

2.1.1. Curvilinear coordinate system adapted to the boundary.

We let $\mathbf{x} = (x_1, x_2, x_3)$ denote the Cartesian coordinates of a point in $\Omega \subset \mathbb{R}^3$. To avoid some technical difficulties of geometry, we assume that the smooth boundary $\Gamma$ is a 2D compact manifold in $\mathbb{R}^3$ having no umbilical points (the two principal curvatures are different at each point on $\Gamma$). Concerning general classes of domains including (isolated) umbilical points, the difficulties and some proper treatments are
explained in, e.g., Section 4 in [43]. Then one can construct a curvilinear system globally on $\Gamma$ in which the metric tensor is diagonal and the coordinate lines at each point are parallel to the principal directions. Such a coordinate system is called the \textit{principal curvature coordinate system}. Inside of a tubular neighborhood $\Omega_\delta$ with a small, but fixed, width $\delta > 0$, we extend the principal curvature coordinates on $\Gamma$ in the direction of $-n$ where $n$ is the outer unit normal on $\Gamma$. As a result, we obtain a triply orthogonal coordinate system $\xi$ in $\mathbb{R}^3_\xi$, such that $\Omega_\delta$ is diffeomorphic to

$$\Omega_{\delta, \xi} := \{ \xi = (\xi', \xi_3) \in \mathbb{R}^3_\xi \mid \xi' = (\xi_1, \xi_2) \in \omega_{\xi'}, \ 0 < \xi_3 < \delta \},$$

for some bounded set $\omega_{\xi'}$ in $\mathbb{R}^2_{\xi'}$. The normal component $\xi_3$ measures the distance from a point in $\Omega_\delta$ to $\Gamma$ and hence we write the boundary $\Gamma$ in the form,

$$\Gamma = \{ \xi \in \mathbb{R}^3_\xi \mid \xi' = (\xi_1, \xi_2) \in \omega_{\xi'}, \ \xi_3 = 0 \}.$$

The need to introduce such tubular domains near the boundary comes from the fact that the boundary layer phenomena are local near the boundary in the direction orthogonal to the boundary but are otherwise nonlocal in the tangential directions.

Using the covariant basis $g_i = \partial x / \partial \xi_i, 1 \leq i \leq 3$, we write the metric tensor of $\xi$,

$$(g_{ij})_{1 \leq i,j \leq 3} := (g_i \cdot g_j)_{1 \leq i,j \leq 3} = \begin{pmatrix}
[1 - \kappa_1(\xi')]\xi_3^2 \tilde{g}_{11}(\xi') & 0 & 0 \\
0 & [1 - \kappa_2(\xi')]\xi_3^2 \tilde{g}_{22}(\xi') & 0 \\
0 & 0 & 1
\end{pmatrix},$$

where $\kappa_i(\xi'), \ i = 1, 2$, is the principal curvature on $\Gamma$, $\tilde{g}_i, \ i = 1, 2$, are the covariant basis of the principal curvature coordinate system on $\Gamma$, and $\tilde{g}_{ii} = \tilde{g}_i \cdot \tilde{g}_i$.

By the choice of a small thickness $\delta > 0$, we have

$$g(\xi) := \det(g_{ij})_{1 \leq i,j \leq 3} > 0 \text{ for all } \xi \text{ in the closure of } \Omega_{\delta, \xi}. \quad (4)$$

We introduce the normalized covariant vectors,

$$e_i = \frac{g_i}{|g_i|}, \quad 1 \leq i \leq 3,$$ 

and set

$$h_i(\xi) = \sqrt{g_{ii}}, \quad i = 1, 2, \quad h(\xi) = \sqrt{g}.$$ 

The function $h(\xi) > 0$ is the magnitude of the Jacobian determinant for the transformation from $x$ in $\Omega_\delta$ to $\xi$ in $\Omega_{\delta, \xi}$. Similarly the function $h(\xi', 0) > 0$ is the magnitude of the Jacobian determinant for the transformation from $x$ on $\Gamma$ to $\xi'$ in $\omega_{\xi'}$.

For a smooth scalar function $v$, defined at least in $\Omega_{\delta}$, we write the gradient of $v$ in the $\xi$ variable,

$$\nabla v = \sum_{i=1}^{2} \frac{1}{h_i} \frac{\partial v}{\partial \xi_i} e_i + \frac{\partial v}{\partial \xi_3} e_3.$$ 

The Laplacian of $v$ is given in the form,

$$\Delta v = S v + L v + \frac{\partial^2 v}{\partial \xi_3^2},$$

where

$$S v = \sum_{i=1,2} \frac{1}{h \partial \xi_i} \left(\frac{h}{h_i^2} \frac{\partial v}{\partial \xi_i}\right), \quad L v = \frac{1}{h} \frac{\partial h}{\partial \xi_3} \frac{\partial v}{\partial \xi_3}.$$
A vector valued function \( v \), defined at least in \( \Omega_\delta \), can be written in the curvilinear system, \( e_1, e_2, e_3 \) as

\[
v = \sum_{i=1}^{3} v_i(\xi) e_i. \tag{10}\]

One can classically express the divergence and curl operators acting on \( v \) in the \( \xi \) variable,

\[
\text{div } v = \frac{1}{h} \sum_{i=1}^{2} \frac{\partial}{\partial \xi_i} \left( \frac{h}{h_i} v_i \right) + \frac{1}{h} \frac{\partial (hv_3)}{\partial \xi_3}, \tag{11}\]

and

\[
\text{curl } v = \frac{h_1}{h} \left[ \frac{\partial v_3}{\partial \xi_2} - \frac{\partial (h_2 v_2)}{\partial \xi_3} \right] e_1 + \frac{h_2}{h} \left[ \frac{\partial (h_1 v_1)}{\partial \xi_3} - \frac{\partial v_3}{\partial \xi_1} \right] e_2 \\
+ \frac{1}{h} \left[ \frac{\partial (h_2 v_2)}{\partial \xi_1} - \frac{\partial (h_1 v_1)}{\partial \xi_2} \right] e_3. \tag{12}\]

The Laplacian of \( v \) is given in the form,

\[
\Delta v = \sum_{i=1}^{3} \left( S^i v + L v_i + \frac{\partial^2 v_i}{\partial \xi^3} \right) e_i, \tag{13}\]

where

\[
\begin{align*}
S^i v &= \left\{ \begin{array}{l}
\text{linear combination of tangential derivatives} \\
\text{of } v^j, 1 \leq j \leq 3, \text{ in } \xi', \text{ up to order 2}
\end{array} \right. \\
L v_i &= \left( \text{proportional to } \frac{\partial v_i}{\partial \xi_3} \right).
\end{align*} \tag{14}\]

**Remark 2.1.** The coefficients of \( S^i, 1 \leq i \leq 3 \), and \( L \) are multiples of \( h, 1/h, h_i, 1/h_i, i = 1, 2 \) and their derivatives. Thanks to (4), all these quantities are well-defined at least in \( \Omega_\delta, \xi \).

Considering smooth vector fields in \( \Omega_\delta \) of the form,

\[
v = \sum_{i=1}^{3} v_i(\xi) e_i, \quad w = \sum_{i=1}^{3} w_i(\xi) e_i,
\]

the covariant derivative of \( w \) in the direction \( v \), which is denoted by \( \nabla_v w \) and gives \( v \cdot \nabla w \) in the Cartesian coordinate system, can be written in the \( \xi \) variable,

\[
\nabla_v w = \sum_{i=1}^{3} \left\{ P^i(v, w) + v_3 \frac{\partial w_i}{\partial \xi_3} + Q^i(v, w) + R^i(v, w) \right\} e_i, \tag{15}\]

where

\[1\] The Laplacian (Laplace-Beltrami operator) of a vector field is defined by the identity \( \Delta v = \nabla (\text{div } v) - \text{curl}(\text{curl } v); \) see, e.g., [20, 87, 6]. We know that other definitions of the Laplacian of a vector, which possess different properties, are used in different contexts.
when a certain symmetry is imposed to the domain $\Omega$. All the analysis below and is valid (and can be made explicit) by using the Lamé replaced by the corresponding expressions

\begin{align}
\mathcal{P}^i(v, w) &= \sum_{j=1}^{2} \frac{1}{h_j} \frac{\partial w_i}{\partial \xi_j} \quad 1 \leq i \leq 3, \\
\mathcal{Q}^i(v, w) &= \begin{cases} \\
\frac{1}{h_1 h_2} \left( \frac{\partial h_i}{\partial \xi_{3-i}} v_i - \frac{\partial h_{3-i}}{\partial \xi_i} v_{3-i} \right) w_{3-i}, & i = 1, 2, \\
- \sum_{j=1}^{2} \frac{1}{h_j} \frac{\partial h_j}{\partial \xi_{3}} v_j w_j, & i = 3,
\end{cases} \\
\mathcal{R}^i(v, w) &= \frac{1}{h_i} \frac{\partial h_i}{\partial \xi_{3}} v_i w_3, \quad i = 1, 2, \quad \mathcal{R}^3(v, w) = 0.
\end{align}

\textbf{Remark 2.2.} The $\mathcal{Q}^i(v, w)$ and $\mathcal{R}^i(v, w)$, $1 \leq i \leq 3$, are related to the Christoffel symbols of the second kind that reflect the twisting effects of the curvilinear system.

2.1.2. \textit{Examples of the curvilinear system with some special geometries.}

We present some examples of the curvilinear coordinates discussed in Section 2.1.1 when a certain symmetry is imposed to the domain $\Omega$. All the analysis below in Sections 2.2 and 2.3 is valid (and can be made explicit) by using the Lamé coefficients $h_i$, $1 \leq i \leq 3$ in Section 2.1.1 replaced by the corresponding expressions in this section.

\textbf{Polar coordinate system}

We consider the domain $\Omega$ in $\mathbb{R}^2$ as a disk with radius $R > 0$,

$$
\Omega = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < R \}. 
$$

(17)

Using the polar coordinates, we construct a curvilinear system adapted to the boundary $\Gamma$ by setting,

$$
x = ((R - \xi_3) \cos \xi_1, (R - \xi_3) \sin \xi_1),
$$

(18)

for $\xi = (\xi_1, \xi_3) \in [0, 2\pi) \times [0, R)$, i.e., all points $x$ in $\Omega \setminus (0, 0)$. Here we suppressed the second tangential variable $\xi_2$ to use $\xi_3$ as the normal variable as appearing in Section 2.1.1.

Differentiating $x$ in (18) with respect to the variable $\xi$, we write the covariant basis,

$$
\begin{align}
g_1 &= \begin{pmatrix} - (R - \xi_3) \sin \xi_1, & (R - \xi_3) \cos \xi_1 \end{pmatrix}, \\
g_3 &= \begin{pmatrix} - \cos \xi_1, & - \sin \xi_1 \end{pmatrix}.
\end{align}
$$

(19)

Using the orthogonality of $\{g_i\}_{i=1,3}$, we write the metric tensor,

$$
(g_{i,j})_{i,j=1,3} =: (g_i \cdot g_j)_{i,j=1,3} = \begin{pmatrix} (R - \xi_3)^2 & 0 \\
0 & 1 \end{pmatrix}.
$$

(20)

Introducing the normalized covariant vectors, $e_i = g_i / |g_i|$, $i = 1, 3$, we find the positive Lamé coefficients,

$$
h_1(\xi_3) = R - \xi_3, \quad h_3 = 1.
$$

(21)

The function $h(\xi_1, \xi_3) := h_1(\xi_3)$ is the magnitude of the Jacobian determinant for the transformation from $x$ to $\xi$, and it is positive for all $\xi$ in $\Omega \setminus (0, 0)$.

\textbf{Cylindrical coordinate system}

The domain $\Omega$ is given as a cylinder in $\mathbb{R}^3$,

$$
\Omega = \{ x \in \mathbb{R}^3 \mid x_1 \in \mathbb{R}, \ x_2^2 + x_3^2 < R \}.
$$

(22)
Using the cylindrical coordinates, we construct a curvilinear system adapted to the boundary \( \Gamma \) by setting,

\[
x = (\xi_2, (R - \xi_3) \cos \xi_1, (R - \xi_3) \sin \xi_1),
\]

(23)

for \( \xi = (\xi_1, \xi_2, \xi_3) \in [0, 2\pi) \times \mathbb{R} \times [0, R) \), i.e., all points \( x \) in \( \Omega \setminus \{x_1 = 0\} \).

Differentiating \( x \) in (23) with respect to \( \xi \), we write the covariant basis,

\[
\begin{align*}
g_1 &= (0, -(R - \xi_3) \sin \xi_1, (R - \xi_3) \cos \xi_1), \\
g_2 &= (1, 0, 0), \\
g_3 &= (0, -\cos \xi_1, -\sin \xi_1). 
\end{align*}
\]

(24)

The metric tensor and positive Lamé coefficients are given in the form,

\[
(g_{i,j})_{1 \leq i,j \leq 3} = (g_i \cdot g_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} (R - \xi_3)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(25)

and

\[
h_1(\xi_3) = R - \xi_3, \quad h_2 = h_3 = 1.
\]

(26)

The function \( h(\xi_1, \xi_3) := h_1(\xi_3) \) is the magnitude of the Jacobian determinant for the transformation from \( x \) to \( \xi \), and it is positive for all \( \xi \) in \( \Omega \setminus \{x_1 = 0\} \).

**Toroidal coordinate system**

We consider the domain \( \Omega \) enclosed by a toroidal surface \( \Gamma \) described by

\[
\Gamma(\xi_1, \xi_2) := ((a + b \cos \xi_1) \cos \xi_2, (a + b \cos \xi_1) \sin \xi_2, b \sin \xi_1), \quad (\xi_1, \xi_2) \in [0, 2\pi)^2,
\]

(27)

for fixed \( 0 < b < a \). Setting \( \xi_3 \) as the distance from a point inside of \( \Omega \) to the toroidal surface \( \Gamma \) (in the direction of \(-n\) on \( \Gamma \)), we construct a curvilinear system \( \xi = (\xi_1, \xi_2, \xi_3) \) via the mapping,

\[
x = ((a + (b - \xi_3) \cos \xi_1) \cos \xi_2, (a + (b - \xi_3) \cos \xi_1) \sin \xi_2, (b - \xi_3) \sin \xi_1),
\]

(28)

for any point \( x \) in the closure of \( \Omega \), except for those along the circle,

\[
C_{\text{sing}.} = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 = a^2 \text{ and } x_3 = 0\}.
\]

(29)

Differentiating \( x \) in (28) with respect to \( \xi \), we find the covariant basis,

\[
\begin{align*}
g_1 &= (b - \xi_3)(- \sin \xi_1 \cos \xi_2, - \sin \xi_1 \sin \xi_2, \cos \xi_1), \\
g_2 &= ((a + b \cos \xi_1) - (\cos \xi_1) \xi_3)(- \sin \xi_2, \cos \xi_2, 0), \\
g_3 &= (\cos \xi_1 \cos \xi_2, \cos \xi_1 \sin \xi_2, \sin \xi_1).
\end{align*}
\]

(30)

Using the orthogonality of \( \{g_i\}_{1 \leq i \leq 3} \), we write the metric tensor and positive Lamé coefficients in the form,

\[
(g_{i,j})_{1 \leq i,j \leq 3} = (g_i \cdot g_j)_{1 \leq i,j \leq 3} = \begin{pmatrix} (b - \xi_3)^2 & 0 & 0 \\ 0 & (a + (b - \xi_3) \cos \xi_1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(31)

and

\[
h_1(\xi_3) = b - \xi_3, \quad h_2(\xi_1, \xi_3) = a + (b - \xi_3) \cos \xi_1, \quad h_3 = 1.
\]

(32)

The function \( h(\xi_1, \xi_3) := h_1(\xi_3)h_2(\xi_1, \xi_3) \) is the magnitude of the Jacobian determinant for the transformation from \( x \) to \( \xi \), and it is positive for all \( \xi \) in \( \Omega \setminus C_{\text{sing}.} \).
Remark 2.3. Spherical coordinate system Considering a ball in $\mathbb{R}^3$, the spherical coordinate system is one of the natural choices to locate a point on the sphere, limiting the ball, but it contains some coordinate singularities at the north and south poles of the sphere and this issue is well-known in atmospheric sciences. To resolve the issue of the pole singularities, one can construct smooth coordinate systems “locally” away from the north and south poles and glue them properly in the “mid-latitude” regions around the equator. We will not discuss this case in details.

2.2. Reaction-diffusion equations in a curved domain.

We consider a reaction-diffusion equation,

$$\begin{cases}
-\varepsilon \Delta u^\varepsilon + u^\varepsilon = f, & \text{in } \Omega, \\
u^\varepsilon = 0, & \text{on } \Gamma,
\end{cases}$$

(33)

where $\Omega$ is a bounded and smooth domain in $\mathbb{R}^3$ as discussed in Section 2.1 and $\varepsilon$ is a small and strictly positive parameter.

We expect that a boundary layer occurs near $\Gamma$ as $\varepsilon$ tends to 0 because the equation (33) formally converges to

$$u^0 = f, \quad \text{in } \Omega,$$

(34)

but $f$ may not vanish on $\Gamma$ in general.

We aim to study the asymptotic behavior of a solution $u^\varepsilon$ to (33) at a small $\varepsilon$ especially when the boundary $\Gamma$ is curved. As we will see below in Theorems 2.1, 2.2, and 2.3, the traditional asymptotic expansion in powers of the small parameter $\varepsilon$ has to be modified by adding terms of order $\varepsilon^{j+1/2}$ in the expansion. In fact these new terms are necessitated by the geometric effect (curvature) of a curved boundary.

2.2.1. Boundary layer analysis at order $\varepsilon^0$.

In this section, we construct an asymptotic expansion at order $\varepsilon^0$ of $u^\varepsilon$, solution of (33), in the form,

$$u^\varepsilon \approx u^0 + \theta^0,$$

(35)

where $u^0 = f$ given in (34) and $\theta^0$ is a corrector function that we will determine below. As we shall see below, the main role of $\theta^0$ is to balance the discrepancy of $u^\varepsilon$ and $u^0$ on the boundary $\Gamma$.

To define a corrector $\theta^0$, we formally insert $\theta^0 \approx u^\varepsilon - u^0$ into the difference of the equations (33) and (34). Introducing a stretched variable $\bar{\xi}_3 = \xi_3 / \sqrt{\varepsilon}$, $\alpha > 0$, and using (8) with (9), we perform the matching asymptotics for the difference equation with respect to a small $\varepsilon$. Then we find that a proper scaling for the stretched variable is

$$\bar{\xi}_3 = \frac{\xi_3}{\sqrt{\varepsilon}},$$

(36)

and that the asymptotic equation for the corrector $\theta^0$ is

$$-\frac{\partial^2 \theta^0}{\partial \bar{\xi}_3^2} + \theta^0 = 0, \quad \text{at least in } \Omega_3.$$

(37)

To make the equation (37) above useful in all of $\Omega$, we first define an exponentially decaying function $\theta^0$ in the half space, $\xi_3 \geq 0$, as a solution of
The explicit expression of $\tilde{\theta}^0$ is given by

$$\tilde{\theta}^0(\xi) = -u^0(\xi', 0) e^{-\frac{\xi_3}{\sqrt{\varepsilon}}}. \quad (39)$$

Using the $\tilde{\theta}^0$ in (39), we define a corrector $\theta^0$ in the form,

$$\theta^0(\xi) := \tilde{\theta}^0(\xi) \sigma(\xi_3), \quad (40)$$

where $\sigma = \sigma(\xi_3)$ is a cut-off function of class $C^\infty$ such that

$$\sigma(\xi_3) = \begin{cases} 1, & 0 \leq \xi_3 \leq \delta/3, \\ 0, & \xi_3 \geq \delta/2, \end{cases} \quad (41)$$

where $\delta > 0$ is the (small) fixed thickness defined in (1).

The equation for $\theta^0$ reads

$$\begin{cases} -\varepsilon \frac{\partial^2 \theta^0}{\partial \xi_3^2} + \theta^0 = -\varepsilon \left( \sigma'' \tilde{\theta}^0 + 2\sigma' \frac{\partial \tilde{\theta}^0}{\partial \xi_3} \right), & \text{in } \Omega, \\ \theta^0 = -u^0, & \text{on } \Gamma. \end{cases} \quad (42)$$

Using (39) and (40), we observe that

$$\text{right-hand side of (42)}_1 = e.s.t., \quad (43)$$

where $e.s.t.$ denotes a term that is exponentially small with respect to a small parameter $\varepsilon$ in any of the usual norms in $\Omega$, e.g., $C^s(\Omega)$ or $H^s(\Omega)$.

To derive some useful estimates on the corrector $\theta^0$, we recall an elementary lemma below:

**Lemma 2.1.** For any $1 \leq p \leq \infty$ and $q \geq 0$, we have

$$\left\| \left( \frac{\xi_3}{\sqrt{\varepsilon}} \right)^q e^{-\frac{\xi_3}{\sqrt{\varepsilon}}} \right\|_{L^p(0, \infty)} \leq \kappa \varepsilon^{-\frac{q}{2p}}. \quad (44)$$

Using (39), (40), and Lemma 2.1, we find that

$$\left\| \left( \frac{\xi_3}{\sqrt{\varepsilon}} \right)^q \frac{\partial^{k+m} \tilde{\theta}^0}{\partial \xi_3^k \partial \xi_3^m} \right\|_{L^p(\Gamma \times \mathbb{R}_+)} \leq \kappa \varepsilon^{-\frac{q}{2p} - \frac{k+m}{2}}, \quad \left\| \left( \frac{\xi_3}{\sqrt{\varepsilon}} \right)^q \frac{\partial^{k+m} \theta^0}{\partial \xi_3^k \partial \xi_3^m} \right\|_{L^p(\Omega)} \leq \kappa \varepsilon^{-\frac{q}{2p} - \frac{k+m}{2}}, \quad (45)$$

for $i = 1$ or 2, $1 \leq p \leq \infty$, $q \geq 0$, and $k, m \geq 0$.

We define the difference between the diffusive solution $\varepsilon u^\varepsilon$ and its asymptotic expansion (35) in the form,

$$w_{\varepsilon,0} := \varepsilon u^\varepsilon - (u^0 + \theta^0). \quad (46)$$

In the theorem below, we state and prove the validity of the asymptotic expansion (35) as well as the convergence of $u^\varepsilon$ to $u^0$:

**Theorem 2.1.** Assuming that the data $f$ belongs to $\{ f \in H^2(\Omega), f|_\Gamma \in W^{2,\infty}(\Gamma) \}$, the difference $w_{\varepsilon,0}$ between the diffusive solution $\varepsilon u^\varepsilon$ and its asymptotic expansion of order $\varepsilon^0$, (see (46)), vanishes as the diffusivity parameter $\varepsilon$ tends to zero in the sense that

$$\| w_{\varepsilon,0} \|_{H^m(\Omega)} \leq \kappa \varepsilon^{-\frac{q}{2p} - \frac{k+m}{2}}, \quad m = 0, 1, \quad (47)$$
for a constant $\kappa$ depending on the data, but independent of $\varepsilon$. Moreover, as $\varepsilon$ tends to zero, $u^\varepsilon$ converges to the limit solution $u^0$ in the sense that

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{3}{4}}.$$

(48)

Furthermore, we have

$$\lim_{\varepsilon \to 0} \left( \frac{\partial u^\varepsilon}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} = \left( \frac{\partial u^0}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} - \left( u^0, \varphi \right)_{L^2(\Gamma)}, \quad \forall \varphi \in C(\Omega),$$

(49)

which expresses the fact that

$$\lim_{\varepsilon \to 0} \frac{\partial u^\varepsilon}{\partial \xi_3} = \frac{\partial u^0}{\partial \xi_3} - u^0(\cdot, 0) \delta_\Gamma^2,$$

(50)

in the sense of weak convergence of bounded measures on $\Omega$.

**Proof.** Using (8), (33), (34), (42), and (43), we write the equation for $w_{\varepsilon,0}$,

$$\begin{cases}
-\varepsilon \Delta w_{\varepsilon,0} + w_{\varepsilon,0} = \varepsilon \Delta u^0 + R_0 + e.s.t., & \text{in } \Omega, \\
 w_{\varepsilon,0} = 0, & \text{on } \Gamma,
\end{cases}$$

(51)

where

$$R_0 = \varepsilon S\theta^0 + \varepsilon L\theta^0.$$  

(52)

Thanks to (9) and (45), we notice that

$$\|R_0\|_{L^2(\Omega)} \leq \kappa \varepsilon \sum_{i=1}^2 \left\| \frac{\partial^2 \theta^0}{\partial \xi_i^2} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\partial \theta^0}{\partial \xi_3} \right\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{3}{4}}.$$ 

(53)

Hence, multiplying (51) by $w_{\varepsilon,0}$, integrating over $\Omega$, and integrating by parts, we find that

$$\varepsilon \|\nabla w_{\varepsilon,0}\|_{L^2(\Omega)}^2 + \|w_{\varepsilon,0}\|_{L^2(\Omega)}^2 \leq \left[ \varepsilon \|\Delta u^0\|_{L^2(\Omega)} + \|R_0\|_{L^2(\Omega)} + e.s.t. \right] \|w_{\varepsilon,0}\|_{L^2(\Omega)}$$

$$\leq \kappa \varepsilon^2 \|\Delta u^0\|_{L^2(\Omega)}^2 + \kappa \|R_0\|_{L^2(\Omega)}^2 + \frac{1}{2} \|w_{\varepsilon,0}\|_{L^2(\Omega)}^2$$

$$\leq \kappa \varepsilon^\frac{3}{4} + \frac{1}{2} \|w_{\varepsilon,0}\|_{L^2(\Omega)}^2.$$ 

(54)

Then we deduce that

$$\|w_{\varepsilon,0}\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{3}{4}}, \quad \|\nabla w_{\varepsilon,0}\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{1}{4}},$$

(55)

and this implies (47).

Thanks to (45), (48) follows from (47) with $m = 0$.

To prove (49), we infer from (47) that

$$\left( \frac{\partial u^\varepsilon}{\partial \xi_3} - \left( \frac{\partial u^0}{\partial \xi_3} + \frac{\partial \theta^0}{\partial \xi_3} \right), \varphi \right)_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{3}{4}}, \quad \forall \varphi \in C(\Omega).$$

(56)

Using (39)-(41), we write

$$\left( \frac{\partial \theta^0}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} = \left( \frac{\partial \tilde{\theta}^0}{\partial \xi_3}, \sigma \right)_{L^2(\Omega)} + \left( \tilde{\theta}^0 \sigma', \varphi \right)_{L^2(\Omega)}.$$ 

(57)

---

Here $\delta_\Gamma$ is used to denote the delta measure on $\Gamma$ and it should not be confused with the ("small") number $\delta$ used at other places in the text.
We observe from (39) and (41) that the second term on the right-hand side of (57) is an e.s.t.. Hence we notice from (56) that
\[
\lim_{\varepsilon \to 0} \left( \frac{\partial u^\varepsilon}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} = \left( \frac{\partial u^0}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} + \lim_{\varepsilon \to 0} \left( \frac{\partial \hat{\varphi}}{\partial \xi_3} \sigma, \varphi \right)_{L^2(\Omega)}, \quad \forall \varphi \in C(\bar{\Omega}), \tag{58}
\]
if the limit on the right-hand side exists.

We introduce the following approximation of the \( \delta \)-measure on \( \mathbb{R} \):
\[
\eta_\varepsilon(x) = \frac{1}{\sqrt{\varepsilon}} \eta \left( \frac{x}{\sqrt{\varepsilon}} \right), \quad \text{where } \eta(x) = \frac{1}{2} e^{-|x|},
\tag{59}
\]
so that \( \|\eta\|_{L^1(\mathbb{R})} = \|\eta_\varepsilon\|_{L^1(\mathbb{R})} = 1 \) for all \( \varepsilon > 0 \). Then we write
\[
\left( \frac{\partial \hat{\varphi}}{\partial \xi_3} \sigma, \varphi \right)_{L^2(\Omega)} = \int_{\omega_{\varepsilon'}} \left( \int_0^\infty \frac{e^{-\varepsilon \frac{\xi_3}{\varepsilon}}}{\sqrt{\varepsilon}} \sigma \varphi h d\xi_3 \right) d\xi' = -\int_{\omega_{\varepsilon'}} u^0(\xi', 0) \left( \int_0^\infty e^{-\varepsilon \frac{\xi_3}{\varepsilon}} \sigma \varphi h d\xi_3 \right) d\xi'.
\]
\[
\int_{\omega_{\varepsilon'}} u^0(\xi', 0) \left( \int_\mathbb{R} \eta_\varepsilon(\xi_3) \sigma(|\xi_3|) \varphi(\xi', |\xi_3|) h(\xi', |\xi_3|) d\xi_3 \right) d\xi'.
\tag{60}
\]
Since \( \eta_\varepsilon \) is an approximation of the \( \delta \)-measure, the inner integral in \( \xi_3 \) converges to \( \sigma \varphi h \) evaluated at \( \xi_3 = 0 \) as \( \varepsilon \) tends to 0. Using this fact and that \( \sigma(0) = 1 \), we deduce from (60) that
\[
\lim_{\varepsilon \to 0} \left( \frac{\partial \hat{\varphi}}{\partial \xi_3} \sigma, \varphi \right)_{L^2(\Omega)} = -\int_{\omega_{\varepsilon'}} u^0(\xi', 0) \varphi(\xi', 0) h(\xi', 0) d\xi' = -(u^0, \varphi)_{L^2(\Gamma)}, \tag{61}
\]
for any \( \varphi \in C(\bar{\Omega}) \). Hence (49) follows from (58) and (61), and now the proof of Theorem 2.1 is complete. \( \square \)

**Remark 2.4.** The weak convergence result (49) in the space of Radon measures is borrowed from [44] in which the authors verified the so-called vorticity accumulation on the boundary of the solutions to the Navier-Stokes at small viscosity. This interesting phenomena was established earlier in [95] by using a method different from that used in [44]. In a closely related article [83], the author proved the equivalence of the vanishing viscosity limit and the vorticity accumulation in a weaker sense, that is, an analogue of (49) with the test functions of class \( C(\bar{\Omega}) \cap H^1 \Omega \).

### 2.2.2. Boundary layer analysis at order \( \varepsilon^{1/2} \): The effect of curvature

Comparing to the case of a domain with flat boundaries, the convergence results in (47) are away from the optimal rates by a factor of \( \varepsilon^{1/4} \); see (63) below. To understand what causes this loss of accuracy, we first notice from (52) and (53) that the bounds in (47) are determined by the \( L^2 \) norm of the term,
\[
\varepsilon L \partial^0,
\tag{62}
\]
which is created solely by the curvature of the boundary. In fact, when the boundary is flat, one can construct an orthogonal coordinates in \( \Omega \) near \( \Gamma \) by taking, at each value of \( \xi_3 \), the identical copy of the surface coordinates on \( \Gamma \). By doing so, the matrix tensor in (3) becomes independent of the normal variable \( \xi_3 \), and hence in the expression (8) of Laplacian, the term \( L \) vanishes thanks to \( \partial h/\partial \xi_3 = 0 \). Moreover, in this flat-boundary case, one can improve the convergence result (47)_{1,2} to
\[
\|u^\varepsilon - (u^0 + \theta^0)\|_{H^m(\Omega)} \leq \kappa \varepsilon^{1 - \frac{m}{2}}, \quad m = 0, 1,
\tag{63}
\]
because the term including $\partial \theta^0 / \partial \xi_3$ vanishes and hence the bound in (53) becomes $\kappa \varepsilon$.

In this section, we will construct a corrector $\theta^{1/2}$ to resolve the effect of curvature, i.e., the (dominant) error in (62), so that we improve the convergence results in (47)$_{1,2}$ to those as in (63) by adding $\theta^{1/2}$ in the asymptotic expansion of $u^\varepsilon$.

Noticing from (9) and (40) that

$$\varepsilon L \theta^0 \approx \sqrt{\varepsilon} e(\xi') e^{-\frac{\xi_3}{\varepsilon}} + \text{e.s.t.},$$

we propose an asymptotic expansion of $u^\varepsilon$ at the order $\varepsilon^{1/2}$ in the form,

$$u^\varepsilon \approx u^0 + \theta^0 + \varepsilon^{1/2} \theta^1.$$ (64)

Here the second corrector $\theta^{1/2}$ will be constructed below as an approximate solution of

$$- \frac{\partial^2 \theta^{1/2}}{\partial \xi_3^2} + \theta^{1/2} = \sqrt{\varepsilon} L \theta^0, \quad \text{at least in } \Omega_\delta.$$ (65)

The natural boundary condition for $\theta^{1/2}$ is

$$\theta^{1/2} = 0, \quad \text{on } \Gamma,$$

because the discrepancy of $u^\varepsilon$ and $u^0$ on $\Gamma$ is already taken care of by introducing $\theta^0$ in the expansion.

For any smooth function $v$, we define

$$L_0 v = \frac{1}{h} \left| \frac{\partial h}{\partial \xi_3} \right|_{\xi_3=0} \frac{\partial v}{\partial \xi_3}.$$ (66)

Then, using the Taylor expansions of $1/h$ and $\partial h / \partial \xi_3$ in $\xi_3$ at $\xi_3 = 0$, we notice that, pointwise:

$$|Lv - L_0 v| = \left| \left( \frac{1}{h} \frac{\partial h}{\partial \xi_3} \right)_{\xi_3=0} \frac{\partial v}{\partial \xi_3} \right| \leq \kappa \varepsilon^{1/2} \left| \xi_3 \frac{\partial v}{\partial \xi_3} \right|. \quad \text{(67)}$$

Using (65) and (66), we define an exponentially decaying function $\bar{\theta}^{1/2}$ in the half space, $\xi_3 \geq 0$, as a solution of

$$\begin{cases} 
- \varepsilon \frac{\partial^2 \bar{\theta}^{1/2}}{\partial \xi_3^2} + \bar{\theta}^{1/2} = \sqrt{\varepsilon} L_0 \bar{\theta}^0, & 0 < \xi_3 < \infty, \\
\bar{\theta}^{1/2} = 0, & \text{at } \xi_3 = 0, \\
\bar{\theta}^{1/2} \to 0, & \text{as } \xi_3 \to \infty.
\end{cases} \quad \text{(68)}$$

Here using (39) and (66), we set

$$\sqrt{\varepsilon} L_0 \bar{\theta}^0 = u^0(\xi', 0) \frac{1}{h} \left| \frac{\partial h}{\partial \xi_3} \right|_{\xi_3=0} e^{-\frac{\xi_3}{\varepsilon}}; \quad \text{(69)}$$

this is the leading order term in small $\varepsilon$ of the right-hand side of (65).

To find the explicit expression of the solution $\bar{\theta}^{1/2}$ for (68), we recall an elementary lemma below:

**Lemma 2.2.** A particular solution of

$$- \alpha^2 \frac{d^2 F}{dx^2}(x) + F(x) = \beta_j \left( \frac{x}{\alpha} \right)^j e^{-\frac{x}{\alpha}}, \quad \alpha \neq 0, \beta_j \in \mathbb{R} \setminus \{0\},$$ (70)
We notice from (70) that the difference $\tilde{w}$ in the sense that

$$F_1(\tilde{x}) = \frac{\beta_3}{2\gamma x^2} \sum_{k=1}^{j+1} \frac{j!}{k!} (2\tilde{x})^k e^{-\tilde{x}}.$$  \hfill (71)

More generally, if the right-hand side of (70) is of the form $P_n(\tilde{x})e^{-\tilde{x}}$ where $P_n(\tilde{x})$ is a polynomial in $\tilde{x}$ of degree $n$, then (70) has a particular solution of the form $P_{n+1}(\tilde{x})e^{-\tilde{x}}$, with $P_n, P_{n+1}$ independent of $\alpha$.

Thanks to Lemma 2.2, we find the solution $\tilde{\theta}^{1/2}$ of (68),

$$\tilde{\theta}^{1/2}(\xi) = \frac{1}{\sqrt{2}} w^0(\xi', 0) \frac{1}{n} \left| \frac{\partial h}{\partial \xi_3} \right| \frac{\xi_3}{\sqrt{\varepsilon}} e^{-\xi_3^{1/2}}.$$  \hfill (72)

Using the cut-off function $\sigma$ in (41), we define the corrector $\theta^{1/2}$ in the form,

$$\theta^{1/2}(\xi) := \tilde{\theta}^{1/2}(\xi)\sigma(\xi_3).$$  \hfill (73)

The equation for $\theta^{1/2}$ reads

$$\left\{ \begin{array}{l}
-\varepsilon \frac{\partial^2 \theta^{1/2}}{\partial \xi_3^2} + \frac{\partial \theta^{1/2}}{\partial \xi_3} = \sqrt{\varepsilon}\sigma L_0 \theta^0 - \varepsilon \left( \sigma'' \theta^{1/2} + 2\sigma' \frac{\partial \theta^{1/2}}{\partial \xi_3} \right), \quad \text{in } \Omega, \\
\theta^{1/2} = 0, \quad \text{on } \Gamma.
\end{array} \right.$$  \hfill (74)

Using (72), (73), and Lemma 2.1, we find that

$$\left\| \frac{\xi_3}{\sqrt{\varepsilon}} \cdot q \frac{\partial^{k+m} \tilde{\theta}^{1/2}}{\partial \xi_3^k \partial \xi_3^m} \right\|_{L^p(\Omega)} \leq \kappa \varepsilon \frac{1}{\varepsilon^{1/2}}, \quad \left\| \frac{\xi_3}{\sqrt{\varepsilon}} \cdot q \frac{\partial^{k+m} \tilde{\theta}^{1/2}}{\partial \xi_3^k \partial \xi_3^m} \right\|_{L^p(\Omega)} \leq \kappa \varepsilon \frac{1}{\varepsilon^{1/2}},$$  \hfill (75)

for $i = 1$ or 2, $1 \leq p \leq \infty$, $q \geq 0$, and $k, m \geq 0$.

We define the difference between $u^\varepsilon$ and the asymptotic expansion at order $\varepsilon^{1/2}$ as

$$w_{\varepsilon, \frac{1}{2}} := u^\varepsilon - (u^0 + \theta^0 + \varepsilon^{1/2} \theta^{1/2}).$$  \hfill (76)

Now we state and prove the validity of the asymptotic expansion (64) which improves (47) in Theorem 2.1:

**Theorem 2.2.** Assuming that the data $f$ belongs to $\{f \in H^2(\Omega), f|_\Gamma \in W^{2,\infty}(\Gamma)\}$, the difference $w_{\varepsilon,1/2}$ between the diffusive solution $u^\varepsilon$ and its asymptotic expansion at order $\varepsilon^{1/2}$ vanishes (or is bounded) as the diffusivity parameter $\varepsilon$ tends to zero in the sense that

$$\left\| w_{\varepsilon, \frac{1}{2}} \right\|_{H^m(\Omega)} \leq \kappa \varepsilon^{1-1/2}, \quad m = 0, 1, 2,$$  \hfill (77)

for a constant $\kappa$ depending on the data, but independent of $\varepsilon$.

**Proof.** We notice from (39), (40), (72), and (73) that

$$\left\{ \begin{array}{l}
-\varepsilon \Delta w_{\varepsilon, \frac{1}{2}} + w_{\varepsilon, \frac{1}{2}} = \varepsilon \Delta u^0 + R_{\frac{1}{2}} + \text{e.s.t.}, \quad \text{in } \Omega, \\
w_{\varepsilon, \frac{1}{2}} = 0, \quad \text{on } \Gamma,
\end{array} \right.$$  \hfill (79)
where

\[ R_+ = \varepsilon S\theta^0 + \varepsilon(L - L_0)\theta^0 + \varepsilon^{\frac{3}{2}} S\theta^1 + \varepsilon^{\frac{3}{2}} L\theta^1. \]  

(80)

Thanks to (9), (45), (67), and (75), we notice that

\[ \|R_+\|_{L^2(\Omega)} \leq \kappa \varepsilon \left\| \frac{\partial^2 \theta^0}{\partial \xi_i^2} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\xi_i}{\sqrt{\varepsilon}} \frac{\partial \theta^0}{\partial \xi_3} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\partial^2 \theta^1}{\partial \xi_i^2} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\partial \theta^1}{\partial \xi_3} \right\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{3}{4}}, \]  

(81)

where \( i = 1 \) or \( 2 \).

Multiplying (79) by \( w_{\varepsilon,1/2} \) and integrating over \( \Omega \), and integrating by parts, we find

\[ \varepsilon \left\| \nabla w_{\varepsilon,1/2} \right\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2} \left\| \Delta u^0 \right\|_{L^2(\Omega)} + \| R_+ \|_{L^2(\Omega)} \leq \kappa \varepsilon \left\| \Delta u^0 \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| R_+ \right\|_{L^2(\Omega)} \leq \kappa \varepsilon^2 + \frac{1}{2} \left\| w_{\varepsilon,1/2} \right\|_{L^2(\Omega)}^2 \]  

(82)

Then we deduce that

\[ \left\| w_{\varepsilon,1/2} \right\|_{L^2(\Omega)} \leq \kappa \varepsilon, \quad \left\| \nabla w_{\varepsilon,1/2} \right\|_{L^2(\Omega)} \leq \kappa \varepsilon^{\frac{3}{4}}, \]  

(83)

and this implies (77) with \( m = 0,1 \).

To verify (77) with \( m = 2 \), we infer from (79), (81), and (83) that

\[ \left\| \Delta w_{\varepsilon,1/2} \right\|_{L^2(\Omega)} \leq 2 \varepsilon \left\| w_{\varepsilon,1/2} \right\|_{L^2(\Omega)} \leq 2 \kappa \varepsilon \left\| \Delta u^0 \right\|_{L^2(\Omega)} + \frac{1}{2} \kappa \varepsilon \left\| R_+ \right\|_{L^2(\Omega)} \leq \kappa. \]  

(84)

Thanks to the regularity theory of elliptic equations, (77) with \( m = 2 \) follows from (84) because of (79).

2.2.3. Asymptotic expansions at arbitrary orders \( \varepsilon^n \) and \( \varepsilon^{n+1/2} \), \( n \geq 0 \).

To extend the convergence results of the diffusive solution \( u^\varepsilon \) to (33) in Theorems 2.1 and 2.2, we construct below asymptotic expansions \( u_n^\varepsilon \) and \( u_{n+1/2}^\varepsilon \) of \( u^\varepsilon \) at arbitrary orders \( n \) and \( n + 1/2 \), \( n \geq 0 \), in the form,

\[
\begin{align*}
  u_n^\varepsilon &= \sum_{j=0}^{n} \left( \varepsilon^j u^j + \varepsilon^{j+\frac{1}{2}} \theta^j \right) + \sum_{j=0}^{n-1} \varepsilon^{j+\frac{1}{2}} \theta^{j+\frac{1}{2}}, \\
  u_{n+\frac{1}{2}}^\varepsilon &= \sum_{j=0}^{n} \left( \varepsilon^j u^j + \varepsilon^{j+\frac{1}{2}} \theta^j \right) + \varepsilon^{j+\frac{1}{2}} \theta^{j+\frac{1}{2}}. 
\end{align*}
\]

(85)

Here the \( u^j \) correspond to the external expansion (outside of the boundary layer) and the correctors \( \theta^j \) and \( \theta^{j+1/2} \) correspond to the inner expansion (inside the boundary layer).

To obtain the external expansion of \( u^\varepsilon \), we formally insert the external expansion \( u^\varepsilon \equiv \sum_{j=0}^{\infty} \varepsilon^j u^j \) into (33) and write

\[ \sum_{j=0}^{\infty} \left( -\varepsilon^{j+1} \Delta u^j + \varepsilon^j u^j \right) \equiv f. \]  

(86)
By matching the terms of the same order \( \varepsilon^j \), we write each \( u^j \) in terms of the data \( f \):

\[
\begin{align*}
    u^0 &= f, \\
    u^j &= \Delta u^{j-1} = \Delta^j f, \quad 1 \leq j \leq n.
\end{align*}
\]

(87)

Note that generally the \( u^j \)'s, \( 0 \leq j \leq n \), do not necessarily vanish on the boundary \( \Gamma \). In fact the discrepancy between \( u^\varepsilon \) and \( \sum_{j=0}^{n} \varepsilon^j u^j \) on \( \Gamma \) creates the boundary layers near \( \Gamma \).

To balance the discrepancy between \( u^\varepsilon \) and the external expansion at order \( n \), we introduce the inner expansion near \( \Gamma \) in the form,

\[
    u^\varepsilon - \sum_{j=0}^{n} \varepsilon^j u^j \cong \sum_{j=0}^{n} \left( \varepsilon^j \theta^j + \varepsilon^{j+\frac{1}{2}} \frac{\theta^{j+\frac{1}{2}}}{2} \right), \quad \text{at least near } \Gamma.
\]

(88)

As we will see below, the corrector \( \theta^j \), \( 0 \leq j \leq n \), balances the discrepancy on \( \Gamma \) between \( u^\varepsilon \) and the proposed external expansion, which is caused by the term \( u^j \).

Then, at each order \( \varepsilon^j \), \( 0 \leq j \leq n \), an additional corrector \( \theta^{j+\frac{1}{2}}/2 \) is introduced in the inner expansion to handle the geometry of the curved boundary. As it appears in Theorems 2.2 and 2.3 below, adding the corrector \( \theta^{j+\frac{1}{2}}/2 \) in the expansion ensures the optimal convergence rate at each order \( \varepsilon^j \), \( 0 \leq j \leq n \).

By matching the terms of the same order \( \varepsilon^j \) on \( \Gamma \), we deduce from (88) the boundary condition for each \( \theta^j \):

\[
    \theta^j = -u^j, \quad \theta^{j+\frac{1}{2}} = 0, \quad \text{on } \Gamma, \ 0 \leq j \leq n.
\]

(89)

To find suitable equations for \( \theta^j \) and \( \theta^{j+\frac{1}{2}} \), \( 0 \leq j \leq n \), we use (33) and (87) as well as (88), and write,

\[
    -\varepsilon \sum_{j=0}^{n} \left( \varepsilon^j \Delta \theta^j + \varepsilon^{j+\frac{1}{2}} \Delta \theta^{j+\frac{1}{2}} \right) + \sum_{j=0}^{\infty} \left( \varepsilon^j \theta^j + \varepsilon^{j+\frac{1}{2}} \frac{\theta^{j+\frac{1}{2}}}{2} \right) \cong 0, \quad \text{at least near } \Gamma.
\]

(90)

Recalling that the size of the boundary layers for the problem (33) is of order \( \varepsilon^{1/2} \), we use below the stretched variable \( \xi_3 = \xi_3/\sqrt{\varepsilon} \) as in (36). To describe the dependency of the Laplacian on the diffusivity parameter \( \varepsilon \), we introduce the Taylor expansion of a smooth function in the normal variable \( \xi_3 \),

\[
    \phi(\xi', \xi_3) = \phi(\xi', \sqrt{\varepsilon} \xi_3) \cong \sum_{j=0}^{\infty} \varepsilon^{j+\frac{1}{2}} \xi_3^j \phi_j, \quad \forall \phi \in C^\infty(\omega_{\xi'} \times [0, \infty)),
\]

(91)

where

\[
    \phi_j := \frac{1}{j!} \frac{\partial^j \phi}{\partial \xi_3^j}(\xi', 0), \quad j \geq 0.
\]

(92)

Using this form of the Taylor expansion for \( h \), \( 1/h \), \( \partial h/\partial \xi_3 \), and \( 1/h_i^2 \), \( i = 1, 2 \), we write the operators \( S \) and \( L \) in (9) as

\[
    S \cong \sum_{j=0}^{\infty} \varepsilon^{j+\frac{1}{2}} \xi_3^j S_{\frac{j}{4}}, \quad L \cong \sum_{j=0}^{\infty} \varepsilon^{j+\frac{1}{2}} \xi_3^j L_{\frac{j}{4}}.
\]

(93)
where

\[
\begin{aligned}
S_j &= \sum_{i=1,2} \sum_{j_1+j_2+j_3=j, (j_1,j_2,j_3) \in \mathbb{N}^3} \left( \frac{1}{h_j} \right)_{j_1} \frac{\partial}{\partial \xi_i} \left\{ (h_j)_{j_2} \left( \frac{1}{h_j} \right)_{j_3} \frac{\partial}{\partial \xi_i} \right\}, \\
L_j &= \sum_{j_1+j_2+j_3=j, (j_1,j_2,j_3) \in \mathbb{N}^3} (\frac{1}{h_j} (h')_{j_2} \frac{\partial}{\partial \xi_3} = \varepsilon^{-\frac{j}{2}} \sum_{j_1+j_2+j_3=j, (j_1,j_2,j_3) \in \mathbb{N}^3} (\frac{1}{h_j} (h')_{j_2} \frac{\partial}{\partial \xi_3}.
\end{aligned}
\]  

(94)

The operators $S_{j/2}$ and $L_{j/2}$, $j \geq 0$, are well-defined if

\[
\partial \Omega \text{ is of class } C^{j+2}.
\]  

(95)

Each $S_{j/2}$, $j \geq 0$, is a tangential differential operator near $\Gamma$ and the $L_{j/2}$ are proportional to $\partial/\partial \xi_3 = \varepsilon^{-1/2} \partial/\partial \xi_3$. Hence the $S_{j/2}$ and $L_{j/2}$ at each $j \geq 0$ are respectively of order $\varepsilon^0$ and $\varepsilon^{-1/2}$ with respect to the small $\varepsilon$.

**Remark 2.5.** We use the stretched variable $\tilde{\xi}_3$ to weight the different terms in the equation (90). Otherwise in the analysis (above and below) we generally revert to the initial variable $\xi_3$.

**Remark 2.6.** As explained in Section 2.2.2, if the boundary of a domain is flat, the operator $L$ in (9) is identically zero and, in this case, the correctors $\theta^{j+1/2}$, $0 \leq j \leq n$, are not required to derive the optimal estimates in Theorems 2.2 and 2.3 below.

Using (9), (93), and (94), we collect all terms of order $\varepsilon^j$ in (90) and find the equation of $\theta^j$ and $\theta^{j+1/2}$, $j \geq 0$:

\[
-\varepsilon \frac{\partial^2 \theta^j + \frac{\partial \theta^j}{\partial \xi_3}}{\partial \xi_3^2} + \theta^{j+\frac{1}{2}} = f^j(\theta), \quad \text{d = 0, 1, at least in } \Omega_\delta,
\]  

(96)

where

\[
\begin{aligned}
f_j^0(\theta) := \sum_{k=0}^{2j-2} \varepsilon^{\frac{k}{2}} \xi_3^k S^k \theta^{j-1-\frac{k}{2}} + \sum_{k=0}^{2j-1} \varepsilon^{\frac{k}{2}} \xi_3^k L^k \theta^{j-\frac{k}{2}}, \\
f_j^{1/2}(\theta) := \sum_{k=0}^{2j-1} \varepsilon^{\frac{k}{2}} \xi_3^k S^k \theta^{j-\frac{k}{2}} + \sum_{k=0}^{2j} \varepsilon^{\frac{k}{2}} \xi_3^k L^k \theta^{j-\frac{k}{2}}.
\end{aligned}
\]  

(97)

The equations above with $n = 0$ are identical to those in (37) and (65).

Modifying the equations (96) and (97), and using the boundary conditions (89), we define the exponentially decaying functions $\tilde{\theta}^j$ and $\tilde{\theta}^{j+1/2}$, $j \geq 0$, as the solutions of

\[
\begin{aligned}
-\varepsilon \frac{\partial^2 \tilde{\theta}^j}{\partial \xi_3^2} + \tilde{\theta}^j &= f_j^0(\tilde{\theta}), \quad 0 < \xi_3 < \infty, \\
\tilde{\theta}^j &= -u^j, \quad \text{at } \xi_3 = 0, \\
\tilde{\theta}^j &\to 0, \quad \text{as } \xi_3 \to \infty,
\end{aligned}
\]  

(98)

and

\[
\begin{aligned}
-\varepsilon \frac{\partial^2 \tilde{\theta}^{j+\frac{1}{2}}}{\partial \xi_3^2} + \tilde{\theta}^{j+\frac{1}{2}} &= f_j^{1/2}(\tilde{\theta}), \quad 0 < \xi_3 < \infty, \\
\tilde{\theta}^{j+\frac{1}{2}} &= 0, \quad \text{at } \xi_3 = 0, \\
\tilde{\theta}^{j+\frac{1}{2}} &\to 0, \quad \text{as } \xi_3 \to \infty.
\end{aligned}
\]  

(99)
\[ f^j_\varepsilon (\bar{\theta}):= \text{(the right-hand side of} \; (97) \text{ with} \; \theta \text{ replaced by} \; \bar{\theta}), \quad d = 0, 1. \quad (100) \]

The equations above with \( j = 0 \) are identical to (38) and (68).

Thanks to Lemma 2.2, we find the structure (and not the explicit expression) of the \( \theta^j \) and \( \theta^{j+1/2} \), \( j \geq 0 \), and this is sufficient for us to perform the error analysis later on:

**Lemma 2.3.** The solutions \( \bar{\theta}^j \) and \( \bar{\theta}^{j+1/2} \), \( j \geq 0 \), of the equations (98) and (99) are of the form,

\[
\bar{\theta}^j (\xi) = P_{2j} \left( \frac{\xi_3}{\sqrt{\varepsilon}} \right) \exp \left( - \frac{\xi_3}{\sqrt{\varepsilon}} \right), \quad \bar{\theta}^{j+1/2} (\xi) = P_{2j+1} \left( \frac{\xi_3}{\sqrt{\varepsilon}} \right) \exp \left( - \frac{\xi_3}{\sqrt{\varepsilon}} \right),
\]

where \( P_k(\xi_3/\sqrt{\varepsilon}) \) is a polynomial of order \( k \) in \( \xi_3/\varepsilon \) whose coefficients are smooth function of \( \xi' \) independent of \( \varepsilon \).

**Proof.** We proceed by induction on \( j \).

Thanks to (39) and (72), we first notice that (101) holds true for \( j = 0 \).

Now, we assume that (101) holds for \( 0 \leq j \leq l - 1 \). Then, to prove that (101)_1 is true when \( j \) is equal to \( l \), we consider the equation (98) with \( j \) replaced by \( l \). With the inductive assumption, we observe that

\[
f^l_\varepsilon (\bar{\theta}) = \sum_{k=0}^{2l-2} \varepsilon^k \xi_3^k S^k_2 \bar{\theta}^{l-1-k} + \sum_{k=0}^{2l-1} \varepsilon^k \xi_3^k L^k_2 \bar{\theta}^{l-2-k} - \frac{3}{2} \frac{\xi_3}{\sqrt{\varepsilon}} \exp \left( - \frac{\xi_3}{\sqrt{\varepsilon}} \right).
\]

Then, using Lemma 2.2, we obtain a particular solution of (98) with \( j \) replaced by \( l \):

\[
\theta^l_p = P_{2l} \left( \frac{\xi_3}{\sqrt{\varepsilon}} \right) \exp \left( - \frac{\xi_3}{\sqrt{\varepsilon}} \right). \quad (103)
\]

Therefore (101)_1 holds true for \( j = l \), since the homogeneous solution of (98) reads

\[
\theta^l_h = -\Delta \theta (\xi', 0) \exp \left( - \frac{\xi_3}{\sqrt{\varepsilon}} \right).
\]

Because (101)_2 can be proved in the same way, the proof is now complete. \( \square \)

Using the cut-off function \( \sigma \) in (41), we now define the correctors \( \theta^j \) and \( \theta^{j+1/2} \), \( 0 \leq j \leq n \), in the form,

\[
\theta^{j+1/2} (\xi) := \bar{\theta}^{j+1/2} (\xi) \sigma(\xi_3), \quad d = 0, 1. \quad (104)
\]

We deduce from Lemma 2.1, (101), and (104) that

\[
\left\| \frac{\xi_3}{\sqrt{\varepsilon}} \frac{\partial^{k+m} \bar{\theta}^{j+1/2}}{\partial \xi_3^k \partial \xi_3^m} \right\|_{L^p(\omega, \Omega)} \leq \kappa \varepsilon^{-\frac{k}{p}} + \frac{\varepsilon}{\sqrt{\varepsilon}}, \quad \left\| \frac{\xi_3}{\sqrt{\varepsilon}} \frac{\partial^{k+m} \bar{\theta}^{j+1/2}}{\partial \xi_3^k \partial \xi_3^m} \right\|_{L^p(\Omega)} \leq \kappa \varepsilon^{-\frac{k}{p}} + \frac{\varepsilon}{\sqrt{\varepsilon}},
\]

for \( d = 1 \) or 2, \( 0 \leq j \leq n \), \( i = 1 \) or 2, \( 1 \leq p \leq \infty \), \( q \geq 0 \), and \( k, m \geq 0 \).

Using (98), (99), (101) and (104), we write the equation for \( \theta^j \) and \( \theta^{j+1/2} \) as

\[
\begin{cases}
-\varepsilon \frac{\partial^2 \theta}{\partial \xi_3^2} + \theta^j = f^j_{\varepsilon} (\theta) + e.s.t., \quad \text{in} \; \Omega, \\
\theta^j = -\omega^j, \quad \text{on} \; \Gamma,
\end{cases}
\]

where \( f^j_{\varepsilon} (\theta) \) is the right-hand side of (97) with \( \theta \) replaced by \( \bar{\theta} \).
where the asymptotic expansion
\[ w \epsilon = \epsilon \theta \epsilon \frac{\partial^{2} \theta^{i+j+\frac{d}{2}}}{\partial z_{3}^{2}} + \theta^{i+j+\frac{d}{2}} = f^{j+\frac{d}{2}}(\theta) + e.s.t., \quad \text{in } \Omega, \]
(107)
\[ \theta^{i+j+\frac{d}{2}} = 0, \quad \text{on } \Gamma. \]

We introduce the remainders at order \( \epsilon^{n} \) and \( \epsilon^{n+1/2}, \) \( n \geq 0, \) in the form,
\[ w_{i,n+\frac{d}{2}} := w_{\epsilon,n+\frac{d}{2}} - u_{\epsilon,n+\frac{d}{2}}, \quad d = 0, 1, \]
(108)
where the asymptotic expansion \( u_{\epsilon,n+d/2} \), \( d = 0, 1, \) of \( u_{\epsilon} \) is given in (85).

Now we state and prove the validity of the asymptotic expansion as a generalization of Theorems 2.1 and 2.2:

**Theorem 2.3.** Assuming that \( f \) belongs to \{ \( f \in H^{2n+2}(\Omega), \ f|_{\Gamma} \in W^{2n+2, \infty}(\Gamma) \} \),
the difference \( w_{\epsilon,n+d/2} \) between the diffusive solution \( u_{\epsilon} \) and its asymptotic expansion of order \( \epsilon^{n+d/2} \), \( d = 0, 1 \) and \( n \geq 0, \)
satisfies
\[ \left\| w_{\epsilon,n+\frac{d}{2}} \right\|_{H^{m}(\Omega)} \leq \kappa \epsilon^{n+\frac{d+4}{2} - \frac{m}{2}}, \quad m = 0, 1, 2, \]
(109)
for a constant \( \kappa \) depending on the data, but independent of \( \epsilon. \)

**Proof.** Using (33), (87), and (108), we write the equations for \( w_{\epsilon,n+d/2}, \) \( d = 0, 1, \)
\( N \geq 0, \)
\[ \left\{ \begin{array}{l}
-\epsilon \Delta w_{\epsilon,n+\frac{d}{2}} + w_{\epsilon,n+\frac{d}{2}} = \epsilon^{n+1} \Delta u_{n} + R_{n+\frac{d}{2}}, \quad \text{in } \Omega, \\
w_{\epsilon,n+\frac{d}{2}} = 0, \quad \text{on } \Gamma,
\end{array} \right. \]
(110)
where
\[ R_{n+\frac{d}{2}} = \sum_{j=0}^{2n+d} (\epsilon \Delta \theta_{j}^{d} - \theta_{j}^{2}), \quad d = 0, 1. \]
(111)

We multiply the equation (110) \( 1 \) by \( w_{\epsilon,n+d/2}, \) integrate over \( \Omega, \) and integrate by parts to find
\[ \epsilon \left\| \nabla w_{\epsilon,n+\frac{d}{2}} \right\|_{L^{2}(\Omega)}^{2} + \frac{1}{2} \left\| w_{\epsilon,n+\frac{d}{2}} \right\|_{L^{2}(\Omega)}^{2} \leq \epsilon^{2n+2} \left\| \Delta u_{n} \right\|_{L^{2}(\Omega)}^{2} + \left\| R_{n+\frac{d}{2}} \right\|_{L^{2}(\Omega)}^{2}. \]
(112)

Using the expression of the Laplacian in (8), (9), and (93), and the equations of the correctors in (106) and (107), we notice that
\[ R_{n} = \sum_{j=0}^{2n-2} \epsilon^{\frac{j+1}{2}} \left\{ S - \sum_{k=0}^{2n-j-2} \xi_{3}^{k} S_{3}^{k} \right\} \theta_{j}^{d} + \epsilon^{n+1} S \theta^{n+1} + \epsilon^{n+1} S \theta^{n} \]
(113)
\[ + \sum_{j=0}^{2n-1} \epsilon^{j+1} \left\{ L - \sum_{k=0}^{2n-j-1} \xi_{3}^{k} L_{3}^{k} \right\} \theta_{j}^{d} + \epsilon^{n+1} L \theta^{n} + e.s.t., \]
and
\[ R_{n+\frac{d}{2}} = \sum_{j=0}^{2n-1} \epsilon^{\frac{j+1}{2}} \left\{ S - \sum_{k=0}^{2n-j-1} \xi_{3}^{k} S_{3}^{k} \right\} \theta_{j}^{d} + \epsilon^{n+1} S \theta^{n} + \epsilon^{n+1} S \theta^{n+1} + \epsilon^{n+1} S \theta^{n+1} \]
(114)
\[ + \sum_{j=0}^{2n} \epsilon^{\frac{j+1}{2}} \left\{ L - \sum_{k=0}^{2n-j} \xi_{3}^{k} L_{3}^{k} \right\} \theta_{j}^{d} + \epsilon^{n+1} L \theta^{n} + e.s.t., \]
We recall that $S$ and the $S_{k/2}$ are tangential differential operators, and that $L$ and the $L_{k/2}$ are proportional to $\partial/\partial \xi_3$. Hence, using (105), we find that

$$
\| R_{n} \|_{L^2(\Omega)} 
\leq \kappa \left\{ \epsilon^{n+\frac{1}{2}} \left\| \sum_{k=0}^{2n-2} \left( \epsilon^{-\frac{3}{2}} \xi_3 \right)^{2n-k-1} S_{n-\frac{k}{2}+\frac{k}{2}} \| \right. + \epsilon^{n+\frac{1}{2}} \left\| S \theta^{n-\frac{1}{2}} \right\|_{L^2(\Omega)} 
+ \epsilon^{n+\frac{1}{2}} \left\| S \theta^{n} \right\|_{L^2(\Omega)} 
+ \epsilon^{n+\frac{1}{2}} \left\| S \theta^{n+1} \right\|_{L^2(\Omega)} \right\} 
\leq \kappa \epsilon^{n+\frac{3}{4}},
$$

(115)

and

$$
\| R_{n+\frac{1}{2}} \|_{L^2(\Omega)} 
\leq \kappa \left\{ \epsilon^{n+\frac{1}{2}} \left\| \sum_{k=0}^{2n-1} \left( \epsilon^{-\frac{3}{2}} \xi_3 \right)^{2n-k} S_{n-\frac{k}{2}+\frac{k}{2}} \right\|_{L^2(\Omega)} + \epsilon^{n+\frac{1}{2}} \left\| S \theta^{n} \right\|_{L^2(\Omega)} 
+ \epsilon^{n+\frac{1}{2}} \left\| S \theta^{n+\frac{1}{2}} \right\|_{L^2(\Omega)} + \epsilon^{n+\frac{1}{2}} \left\| \sum_{k=0}^{2n-1} \left( \epsilon^{-\frac{3}{2}} \xi_3 \right)^{2n-k+1} L_{n-\frac{k}{2}+\frac{k}{2}} \right\|_{L^2(\Omega)} \right\} 
\leq \kappa \epsilon^{n+\frac{3}{4}},
$$

(116)

Then (109) with $m = 0, 1$ follows from (112), (115), and (116).

To verify (109) with $m = 2$, we infer from (109) with $m = 0, 1$, (110)1, (115), and (116) that

$$
\| \Delta w_{\varepsilon, n+\frac{1}{2}} \|_{L^2(\Omega)} \leq \epsilon^{-1} \| w_{\varepsilon, n+\frac{1}{2}} \|_{L^2(\Omega)} + \epsilon^{n} \| \Delta u^{n} \|_{L^2(\Omega)} + \epsilon^{-1} \| R_{n+\frac{1}{2}} \|_{L^2(\Omega)} \leq \kappa \epsilon^{n+\frac{4}{4}},
$$

(117)

for $d = 0, 1$. Thanks to the regularity theory of elliptic equations, (109) with $m = 2$ follows from (117) and (110)2.

2.3. **Parabolic equations in a curved domain.**

We consider the heat equation in a bounded smooth domain $\Omega$ of $\mathbb{R}^3$,

$$
\begin{cases}
\frac{\partial u^\varepsilon}{\partial t} - \varepsilon \Delta u^\varepsilon = f, & \text{in } \Omega \times (0, T), \\
\quad \quad u^\varepsilon = 0, & \text{on } \Gamma \times (0, T), \\
\quad \quad u^\varepsilon|_{t=0} = u_0, & \text{in } \Omega,
\end{cases}
$$

(118)

where $f$ and $u_0$ are given smooth data, $T > 0$ is an arbitrary but fixed time, and $\varepsilon$ is a small strictly positive diffusivity parameter.

It is well-known that the solution to (118) at small $\varepsilon > 0$ produces a large gradient near the boundary $\Gamma$ when the data $u_0$ and $f$ do not vanish on the boundary; see the equation (122) below as for the formal limit of $u^\varepsilon$ at $\varepsilon = 0$. In this section, we
study the asymptotic behavior of the solutions of (118) with respect to the small parameter $\varepsilon > 0$. Using the methodology introduced in Section 2.2, we construct below the asymptotic expansion for $u^\varepsilon$ as the sum of the inner and outer expansions, which gives a complete structural information of $u^\varepsilon$ in powers of $\varepsilon$.

To explain the basis of the boundary layer analysis for (118), we assume in Sections 2.3.1 and 2.3.2 that the smooth initial data is well-prepared,

$$u_0 = 0, \quad \text{on } \Gamma,$$

and construct an asymptotic expansion of $u^\varepsilon$ at an arbitrary order $\varepsilon^n$ and $\varepsilon^{n+1/2}$, $n \geq 0$.

When the initial data is ill-prepared, that is, $u_0 \neq 0$, on $\Gamma$,

it is well-known that the so-called initial layer is impulsively created at the initial time $t = 0$. This interesting phenomenon will be discussed separately in Section 2.3.3.

2.3.1. Boundary layer analysis at orders $\varepsilon^0$ and $\varepsilon^{1/2}$.

In this section, we propose an asymptotic expansion of $u^\varepsilon$ solution of (118) in the form,

$$u^\varepsilon \approx u^0 + \theta^0 + \varepsilon^{1/2} \theta^{1/2},$$

where the formal limit $u^0$ of $u^\varepsilon$ at $\varepsilon = 0$ and the two corrector functions $\theta^0$ and $\theta^{1/2}$ will be determined below.

The limit $u^0$ is defined as the solution of equation (118) with $\varepsilon = 0$:

$$\frac{\partial u^0}{\partial t} = f, \quad \text{in } \Omega \times (0,T), \quad u^0|_{t=0} = u_0, \quad \text{in } \Omega.$$ (122)

Integrating (122) in time, we find

$$u^0(x,t) = u_0(x) + \int_0^t f(x,s) \, ds;$$ (123)

$u^0$ belongs to $C^{k+1}([0,T];H^m(\Omega))$ for any $T > 0$ and $k,m \geq 0$, provided

$$u_0 \in H^m(\Omega), \quad f \in C^k([0,T];H^m(\Omega)).$$

Thanks to the consistency condition (119) on the initial data, we infer from (123) that

$$u^0|_{t=0} = 0, \quad \text{on } \Gamma.$$ (124)

Hence the boundary values of $u^\varepsilon$ and $u^0$ agree as 0 at time $t = 0$. However, for any $t > 0$, we infer from (118) and (123) that

$$u^\varepsilon - u^0 = -u^0 \neq 0, \quad \text{on } \Gamma \text{ (in general)}.$$ (125)

In fact, this discrepancy of $u^\varepsilon$ and $u^0$ on the boundary creates the boundary layers near $\Gamma$, and it necessitates the first corrector $\theta^0$ in the expansion (121) that satisfies,

$$\theta^0 = -u^0, \quad \text{on } \Gamma \times (0,T).$$ (126)

The main role of the corrector $\theta^0$ is to balance the difference $u^\varepsilon - u^0$ on and near the boundary. Then, to manage the geometric effect of a curved boundary, we add the second corrector $\theta^{1/2}$ in the expansion (121) that satisfies the boundary condition,

$$\frac{\partial \theta^{1/2}}{\partial n} = 0, \quad \text{on } \Gamma \times (0,T).$$ (127)
Since the initial data of $u^\varepsilon$ and $u^0$ are the same as $u_0$, it is natural to impose the zero initial condition for both $\theta^0$ and $\theta^{1/2}$:

$$\theta^2 \big|_{t=0} = 0, \quad d = 0, 1, \quad (128)$$

To find an equation for $\theta^0 (\equiv u^\varepsilon - u^0)$, we perform the matching asymptotics for the difference of (118) and (122) with respect to a small diffusivity parameter $\varepsilon > 0$. Using the curvilinear coordinates $\xi$ of Section 2.1.1, we find that a proper scaling for the stretched variable is $\xi_3 / \sqrt{\varepsilon}$, and that an asymptotic equation for $\theta^0$ with respect to $\varepsilon$ is

$$\frac{\partial \theta^0}{\partial t} - \varepsilon \frac{\partial^2 \theta^0}{\partial \xi_3^2} = 0, \quad \text{at least in } \Omega_3 \times (0, T). \quad (129)$$

This process is exactly the same as what we did for the reaction-diffusion equation to obtain (37).

Using (8), (9), and (93), we notice that

$$\Delta (u^\varepsilon - u^0 - \theta^0) \equiv L_0 \theta^0 + l.o.t., \quad (130)$$

with respect to $\varepsilon$. (The operator $L_0$ is identical to that in (66)). Hence, following the methodology in Section 2.2.2, we find an equation for $\sqrt{\varepsilon} \theta^{1/2} (\equiv u^\varepsilon - u^0 - \theta^0)$ as

$$\frac{\partial \theta^\frac{1}{2}}{\partial t} - \varepsilon \frac{\partial^2 \theta^\frac{1}{2}}{\partial \xi_3^2} = \sqrt{\varepsilon} L_0 \theta^0, \quad \text{at least in } \Omega_3 \times (0, T). \quad (131)$$

Now, using (126), (127), (128), (129), and (131), we define the approximate correctors $\bar{\theta}^0$ and $\bar{\theta}^\frac{1}{2}$ as solutions to the heat equations in the half-space, $\xi_3 \geq 0$,

$$\left\{ \begin{array}{l}
\frac{\partial \bar{\theta}^0}{\partial t} - \varepsilon \frac{\partial^2 \bar{\theta}^0}{\partial \xi_3^2} = 0, \quad \xi_3, t > 0, \\
\bar{\theta}^0 = -u^0, \quad \text{at } \xi_3 = 0, \\
\bar{\theta}^0 \to 0, \quad \text{as } \xi_3 \to \infty, \\
\bar{\theta}^0 \big|_{t=0} = 0, \quad \xi_3 > 0,
\end{array} \right. \quad (132)$$

and

$$\left\{ \begin{array}{l}
\frac{\partial \bar{\theta}^\frac{1}{2}}{\partial t} - \varepsilon \frac{\partial^2 \bar{\theta}^\frac{1}{2}}{\partial \xi_3^2} = \sqrt{\varepsilon} L_0 \bar{\theta}^0, \quad \xi_3, t > 0, \\
\bar{\theta}^\frac{1}{2} = 0, \quad \text{at } \xi_3 = 0, \\
\bar{\theta}^\frac{1}{2} \to 0, \quad \text{as } \xi_3 \to \infty, \\
\bar{\theta}^\frac{1}{2} \big|_{t=0} = 0, \quad \xi_3 > 0.
\end{array} \right. \quad (133)$$

From, e.g., [11], we recall that the explicit expression of $\bar{\theta}^0$ (when the initial data is well-prepared to satisfy (119)) is given by

$$\bar{\theta}^0 (\xi, t) = -2 \int_0^t \frac{\partial u_0}{\partial t} (\xi', 0, s) \operatorname{erfc} \left( \frac{\xi_3}{\sqrt{2\varepsilon(t-s)}} \right) ds$$

$$= -2 \int_0^t f(\xi', 0, s) \operatorname{erfc} \left( \frac{\xi_3}{\sqrt{2\varepsilon(t-s)}} \right) ds$$

$$= -2 \int_0^t f(\xi', 0, t-s) \operatorname{erfc} \left( \frac{\xi_3}{\sqrt{2\varepsilon s}} \right) ds, \quad (134)$$
where the complementary error function \( \text{erfc}(\cdot) \) on \( \mathbb{R}_+ \) is defined by

\[
\text{erfc}(z) := \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-y^2/2} dy,
\] (135)

so that

\[
\text{erfc}(0) = \frac{1}{2}, \quad \text{erfc}(\infty) = 0.
\] (136)

The approximate corrector \( \theta^{1/2} \) is given in the form,

\[
\theta^{\frac{1}{2}}(\xi, t) = J_+ - J_-, \quad \text{(137)}
\]

where

\[
J_\pm(\xi, t) = \int_0^t \int_0^\infty \frac{\partial}{\partial \xi_3} \left\{ \text{erfc} \left( \frac{\xi_3 \pm \eta}{\sqrt{2(\xi - s)}} \right) \right\} \sqrt{\varepsilon}(L_0\bar{\theta}^0)(\xi', \eta, s) d\eta ds.
\] (138)

Using the cut-off function \( \sigma \) in (41) and the approximate correctors above, we define the correctors \( \theta^0 \) and \( \theta^{1/2} \) in the form,

\[
\theta^d(\xi, t) := \theta^{\frac{d}{2}}(\xi, t)\sigma(\xi_3), \quad d = 0, 1,
\] (139)

which are functions well-defined in \( \overline{\Omega} \times [0, T] \).

To derive some estimates on the correctors, we first state and prove the pointwise estimates on the complementary error function:

**Lemma 2.4.** For any \((\xi_3, t)\) in \(\mathbb{R}^2_+\), we have

\[
\frac{\partial^m}{\partial \xi_3^m} \left\{ \text{erfc} \left( \frac{\xi_3}{\sqrt{2\varepsilon t}} \right) \right\} \leq \begin{cases}
\kappa \exp \left( -\frac{\xi_3^2}{4\varepsilon t} \right), & m = 0, \\
\kappa (\varepsilon t)^{-m+\frac{1}{2}} \xi_3^{m-1} \exp \left( -\frac{\xi_3^2}{4\varepsilon t} \right), & m = 1, 2, \\
\kappa \varepsilon^{-m+\frac{1}{2}} (1 + t^{-m+\frac{1}{2}})(1 + \xi_3^{m-1}) \exp \left( -\frac{\xi_3^2}{4\varepsilon t} \right), & m \geq 3.
\end{cases}
\] (140)

for a constant \(\kappa\) independent of \(\varepsilon\).

**Proof.** Using polar coordinates, we notice that

\[
(\text{erfc}(z))^2 = \frac{1}{2\pi} \int_z^\infty \int_2^\infty e^{-(y_1^2+y_2^2)/2} dy_1 dy_2 \leq \frac{1}{4} \int_\sqrt{2z}^\infty e^{-r^2/2} r dr \leq \frac{1}{4} e^{-z^2},
\] (141)

and hence (140) follows for \(m = 0\).

Differentiating (135) with respect to \(\xi_3\), the left hand side of (140) is equal to

\[
-\frac{1}{2\sqrt{\pi}} (\varepsilon t)^{-\frac{1}{2}} \exp \left(-\frac{\xi_3^2}{4\varepsilon t}\right), \quad \text{for } m = 1, \quad \text{and} \quad \frac{1}{4\sqrt{\pi}} (\varepsilon t)^{-\frac{3}{2}} \xi_3 \exp \left(-\frac{\xi_3^2}{4\varepsilon t}\right), \quad \text{for } m = 2,
\] (142)

and then (140) immediately follows for \(m = 1, 2\).

Thanks to the Leibnitz formula, we differentiate (142)_2 \((m - 2)\)-times \((m \geq 3)\) in \(\xi_3\) and write

\[
\frac{\partial^m}{\partial \xi_3^m} \left\{ \text{erfc} \left( \frac{\xi_3}{\sqrt{2\varepsilon t}} \right) \right\} = -\frac{1}{\sqrt{\pi}} (-2)^{-m+1} (\varepsilon t)^{-m+\frac{1}{2}} \xi_3^{m-1} \exp \left(-\frac{\xi_3^2}{4\varepsilon t}\right) + r_{0,m}(\xi_3, t),
\] (143)
where \( r_{0,m} \), \( m \geq 3 \), is given in the form,

\[
\begin{align*}
\left\{
\begin{array}{l}
\forall \ell \leq 2, \\
\forall \ell \leq 2, \\
\ell + 2
\end{array}
\right.
\end{align*}
\]

(144)

for some strictly positive integers \( a_{i,m}, 0 \leq i \leq n - 2 \). Using (144), we bound the lower order term \( r_{0,m} \) with respect to \( \varepsilon \),

\[
|r_{0,m}(\xi, t)| \leq \kappa e^{-m+2}(1 + t^{-m+2})(1 + \varepsilon^{m-3}) \exp \left( -\frac{\varepsilon^3}{4\varepsilon t} \right),
\]

(145)

and, from (143) and (145), we obtain (140) for \( m \geq 3 \).

Now we state and prove some pointwise estimates on \( \tilde{\theta}^{d/2}, d = 0, 1 \):\

**Lemma 2.5.** Assuming that \( u_0 \) satisfies the compatibility condition (119) and \( f | T \) belongs to \( C^1([0, T]; W^{k, \infty}(\Gamma)) \), the approximate corrector \( \tilde{\theta}^0 \) satisfies the pointwise estimate,

\[
\frac{\partial^{l+k+m} \tilde{\theta}^0}{\partial t^l \partial \xi_3^k \partial \xi_3^m} \leq \kappa_T \varepsilon^{-l-m+\frac{3}{2}} \exp \left( -\frac{\varepsilon^3}{4\varepsilon t} \right), \quad (\xi, t) \in \omega_{\xi} \times \mathbb{R} \times (0, T),
\]

(146)

for \( \ell = 0, k \geq 0 \), and \( 0 \leq m \leq 3 \), or \( \ell = 1, k \geq 0 \), and \( m = 0, 1 \), and

\[
\frac{\partial^{l+k+m} \tilde{\theta}^0}{\partial t^l \partial \xi_3^k \partial \xi_3^m} \leq \kappa_{T, \delta} \varepsilon^{-l-m+\frac{3}{2}} \int_0^t (1 + s^{-2l-m+\frac{3}{2}}) \exp \left( -\frac{\varepsilon^3}{4\varepsilon s} \right) ds,
\]

(147)

for \( (\xi, t) \in \omega_{\xi} \times (0, \delta) \times (0, T), \ell = 0, k \geq 0 \), and \( m \geq 4 \), \( \ell = 1, k \geq 0 \), and \( m \geq 2 \), or \( \ell = 2 \), and \( k, m \geq 0 \). The constant \( \kappa_T \) (or \( \kappa_{T, \delta} \)) depends on \( T \) (or \( T \) and \( \delta \)) and the other data, but is independent of \( \varepsilon \).

**Proof.** Using (134), we write, for \( i = 1, 2 \) and \( k, m \geq 0 \),

\[
\frac{\partial^{k+m} \tilde{\theta}_i^0}{\partial \xi_3^k \partial \xi_3^m} = -2 \int_0^t \frac{\partial f}{\partial \xi_3^i}(\xi', 0, t-s) \frac{\partial^m}{\partial \xi_3^m} \left\{ \text{erfc} \left( \frac{\xi_3}{\sqrt{2\varepsilon s}} \right) \right\} ds.
\]

(148)

Then (146) with \( \ell = 0, k \geq 0 \), and \( m = 0, 1 \) follows from (140) because \( s^{-1/2} \) is integrable over \( (0, T) \). Using (132), we write

\[
\frac{\partial^{k+m+2} \tilde{\theta}_i^0}{\partial \xi_3^k \partial \xi_3^m} = \varepsilon^{-1} \frac{\partial^{k+m+1} \tilde{\theta}_i^0}{\partial t \partial \xi_3^k \partial \xi_3^m}
\]

\[
= -2\varepsilon^{-1} \frac{\partial f}{\partial \xi_3^i}(\xi', 0, 0) \frac{\partial^m}{\partial \xi_3^m} \left\{ \text{erfc} \left( \frac{\xi_3}{\sqrt{2\varepsilon t}} \right) \right\}
\]

(149)

Then (146) with \( \ell = 0, k \geq 0 \), and \( m = 2, 3 \) follows from (140), and hence we obtain (146) with \( \ell = 1, k \geq 0 \), and \( m = 0, 1 \) as well, using the heat equation (132).
We infer from (140) and (148) that
\[
\left| \frac{\partial^{k+m} \bar{\theta}^0}{\partial \xi_i^k \partial \xi_3^m} \right| \leq \kappa_T \varepsilon^{-m+\frac{1}{2}} \int_0^t (1 + (s')^{-m+\frac{1}{2}})(1 + \varepsilon_3^{m-1}) \exp \left( -\frac{\varepsilon_3^2}{4\varepsilon s} \right) ds' \\
\leq \kappa_T (1 + \delta^{m-1}) \varepsilon^{-m+\frac{1}{2}} \int_0^t (1 + (s')^{-m+\frac{1}{2}}) \exp \left( -\frac{\varepsilon_3^2}{4\varepsilon s} \right) ds', \quad m \geq 4,
\]
and this implies (147) with \( \ell = 0, k \geq 0, \) and \( m \geq 4. \)

Using the heat equation (132), one can prove (147) with \( \ell = 1, k \geq 0, \) and \( m \geq 2 \) or \( \ell \geq 2 \) and \( k, m \geq 0 \) by applying the same method as for (150).

**Lemma 2.6.** Assuming that \( u_0 \) satisfies the compatibility condition (119) and \( f|_{\Gamma} \) belongs to \( C^1([0,T];W^{k,\infty}(\Gamma)) \), the approximate corrector \( \bar{\theta}^{1/2} \) satisfies the pointwise estimate,
\[
\left| \frac{\partial^{k+m} \bar{\theta}^\frac{1}{2}}{\partial \xi_i^k \partial \xi_3^m} \right| \leq \kappa_T \varepsilon^{-m+\frac{1}{2}} \exp \left( -\frac{\varepsilon_3^2}{8(1+m)\varepsilon t} \right), \quad (\xi,t) \in \omega_{\xi'} \times \mathbb{R} \times (0,T),
\]
for \( k \geq 0 \) and \( m = 0, 1, \) and
\[
\left| \frac{\partial^{\ell+k+m} \bar{\theta}^\frac{1}{2}}{\partial \xi_i^\ell \partial \xi_3^m} \right| \leq \kappa_{T,\delta} \varepsilon^{-\ell-m+\frac{1}{2}} \int_0^t (1 + s^{-2\ell-m+\frac{1}{2}}) \exp \left( -\frac{\varepsilon_3^2}{8s} \right) ds,
\]
for \( (\xi,t) \in \omega_{\xi'} \times (0,\delta) \times (0,T), \) \( \ell = 0, k \geq 0, \) and \( m \geq 2, \) or \( \ell \geq 1 \) and \( k, m \geq 0. \)
The constant \( \kappa_T \) (or \( \kappa_{T,\delta} \)) depends on \( T \) (or \( T \) and \( \delta \)) and the other data, but is independent of \( \varepsilon. \)

**Proof.** Using (137) and (138), we write
\[
\frac{\partial^{k+m} \bar{\theta}^\frac{1}{2}}{\partial \xi_i^k \partial \xi_3^m}(\xi',\xi_3,t) = \frac{\partial^{k+m} J_+}{\partial \xi_i^k \partial \xi_3^m} + \frac{\partial^{k+m} J_-}{\partial \xi_i^k \partial \xi_3^m}, \quad i = 1, 2, k, m \geq 0,
\]
where
\[
\frac{\partial^{k+m} J_{\pm}}{\partial \xi_i^k \partial \xi_3^m} = \int_0^t \int_0^\infty \frac{\partial^{m+1} J_{\pm}}{\partial \xi_i^k \partial \xi_3^m} \left\{ \text{erfc} \left( -\frac{\xi_3^{\pm} + \eta}{\sqrt{2\varepsilon(t-s)}} \right) \right\} \frac{\partial^k (\sqrt{\varepsilon L_0} \bar{\theta}^0)}{\partial \xi_i^k}(\xi',\eta,s) d\eta ds.
\]
Using (66) and (146), we observe that
\[
\left| \frac{\partial^k (\sqrt{\varepsilon L_0} \bar{\theta}^0)}{\partial \xi_i^k}(\xi',\eta,s) \right| \leq \kappa \sum_{r=0}^k \left| \frac{\partial^{r+1} \bar{\theta}^0}{\partial \xi_i^r \partial \xi_3^r}(\xi',\eta,s) \right| \leq \kappa_T \exp \left( -\frac{\eta^2}{4\varepsilon s} \right),
\]
for each \( (\xi',\eta,s) \in \omega_{\xi'} \times \mathbb{R} \times (0,T), \) and \( i = 1, 2. \)

Now, concerning (151), we only show below the case when \( m = 1 \) because (151) with \( m = 0 \) can be verified in a similar but easier way.

To estimate \( \partial^{k+1} J_+/\partial \xi_i^k \partial \xi_3 \) pointwise, we write this term as the sum of two integrals \( \partial^{k+1} J_+/\partial \xi_i^k \partial \xi_3 \) on \( (0,t/2) \) and \( \partial^{k+1} J_2/\partial \xi_i^k \partial \xi_3 \) on \( (t/2,t) \), and estimate them below separately:
We first estimate the more problematic integral on \((t/2, t)\) by using (140), (155), and the Schwarz inequality,
\[
\left| \frac{\partial^{k+1} J^2}{\partial \xi^k_1 \partial \xi_3} \right| \leq \kappa_T \varepsilon^{-\frac{1}{2}} \int_{t/2}^t \int_0^\infty |\xi_3 - \eta| \exp \left( -\frac{(\xi_3 - \eta)^2}{4 \varepsilon (t - s)} \right) \exp \left( -\frac{\eta^2}{2 \varepsilon s} \right) d\eta ds
\leq \kappa_T \varepsilon^{-\frac{1}{2}} \int_{t/2}^t (t - s)^{-1} \left[ \int_0^\infty \frac{|\xi_3 - \eta|^2}{t - s} \exp \left( -\frac{(\xi_3 - \eta)^2}{4 \varepsilon (t - s)} \right) d\eta \right]^\frac{1}{2} \left[ \int_0^\infty \exp \left( -\frac{(\xi_3 - \eta)^2}{4 \varepsilon (t - s)} \right) \exp \left( -\frac{\eta^2}{2 \varepsilon s} \right) d\eta \right]^\frac{1}{2} ds.
\]
(156)

Setting \(\eta' = (\eta - \xi_3)/\sqrt{2 \varepsilon (t - s)}\), we observe that
\[
\int_0^\infty \frac{|\xi_3 - \eta|^2}{t - s} \exp \left( -\frac{(\xi_3 - \eta)^2}{4 \varepsilon (t - s)} \right) d\eta \leq (2\varepsilon)^{\frac{1}{2}} \sqrt{t - s} \int_{-\infty}^\infty (\eta')^2 e^{(\eta')^2/2} d\eta'
\leq \kappa \varepsilon^{\frac{1}{2}} \sqrt{t - s}.
\]
(157)

Using the fact that \(t - s < s\) for \(t/2 < s < t\) and setting \(\eta' = (\eta - \xi_3/2)/\sqrt{\varepsilon s}\), we find
\[
\int_0^\infty \exp \left( -\frac{(\xi_3 - \eta)^2}{4 \varepsilon (t - s)} - \frac{\eta^2}{2 \varepsilon s} \right) d\eta \leq \int_0^\infty \exp \left( -\frac{(\xi_3 - \eta)^2}{4 \varepsilon s} \right) \exp \left( -\frac{\eta^2}{2 \varepsilon s} \right) d\eta
\leq \varepsilon^{\frac{1}{4}} \sqrt{\varepsilon s} \exp \left( -\frac{\xi_3^2}{8 \varepsilon s} \right).
\]
(158)

Combining (156)-(158), we see that
\[
\left| \frac{\partial^{k+1} J^2}{\partial \xi^k_1 \partial \xi_3} \right| \leq \kappa_T \varepsilon^{-\frac{1}{2}} \int_{t/2}^t (t - s)^{-\frac{1}{4}} \exp \left( -\frac{\xi_3^2}{16 \varepsilon s} \right) ds
\leq \kappa_T \varepsilon^{-\frac{1}{2}} \exp \left( -\frac{\xi_3^2}{16 \varepsilon t} \right) \int_{t/2}^t (t - s)^{-\frac{1}{4}} ds
\leq \kappa_T \varepsilon^{-\frac{1}{2}} \exp \left( -\frac{\xi_3^2}{16 \varepsilon t} \right).
\]
(159)

The other integral \(\partial^{k+1} J^1_\perp/\partial \xi^k_1 \partial \xi_3\) on \((0, t/2)\) satisfies the estimate (156) with the interval \((t/2, t)\) replaced by \((0, t/2)\). Then, since \((t - s)^{-3/2}\) is bounded from below and above by \((t/2)^{-3/2}\) and \(t^{-3/2}\) on \((0, t/2)\), it is easy to see that \(\left| \partial^{k+1} J^1_\perp/\partial \xi^k_1 \partial \xi_3 \right|\) is also bounded by the right-hand side of (159), and we deduce that
\[
\left| \frac{\partial^{k+1} J^1_\perp}{\partial \xi^k_1 \partial \xi_3} \right| \leq \kappa_T \varepsilon^{-\frac{1}{2}} \exp \left( -\frac{\xi_3^2}{16 \varepsilon t} \right).
\]
(160)

With the same (but easier) proof for the term \(\partial^{k+1} J^1_\perp/\partial \xi^k_1 \partial \xi_3\), we obtain (151) with \(m = 1\).
To show (152) with \( \ell = 0, k \geq 0, \) and \( m \geq 2, \) we use (2.4), (153), and (155), and find that
\[
\frac{\partial^{k+m}f}{\partial x^k \partial \xi^m} \leq \kappa_T \varepsilon^{-m-\frac{1}{2}} \int_0^t \int_0^{\infty} \{1+(t-s)^{-m-\frac{1}{2}}\} \{1+|\xi_3|+\eta|^m\} \exp\left(-\frac{(\xi_3+\eta)^2}{4\varepsilon(t-s)} - \frac{\eta^2}{4\varepsilon s}\right) \, dpds
\]
(161)
\[
\leq (\text{using the analog of (158)}) \leq \kappa_{T,\delta} \varepsilon^{-m-\frac{1}{2}} \int_0^t \{1+(t-s)^{-m-\frac{1}{2}}\} \exp\left(-\frac{\xi_3^2}{8\varepsilon(t-s)}\right) \, ds.
\]
Thus (152) with \( \ell = 0, k \geq 0, \) and \( m \geq 2 \) follows.

For (152) with \( \ell \geq 1 \) and \( k, m \geq 0, \) we use (133) and write
\[
\frac{\partial^{k+m+1} \tilde{\theta}^+}{\partial \xi_t \partial \xi^k \partial \xi^m} = \varepsilon \frac{\partial^{k+m+2} \tilde{\theta}^+}{\partial \xi_t \partial \xi^k \partial \xi^{m+2}} + \frac{\partial^{k+m}(\sqrt{L_0} \tilde{\theta})}{\partial \xi_t \partial \xi^k \partial \xi^m},
\]
(162)
and more generally,
\[
\frac{\partial^{\ell+k+m} \tilde{\theta}^+}{\partial t^\ell \partial \xi^k \partial \xi^m} = \varepsilon^{\ell} \frac{\partial^{2\ell+k+m} \tilde{\theta}^+}{\partial \xi_t^{2\ell} \partial \xi^k \partial \xi^{m} + \sum_{i=0}^{\ell-1} \varepsilon^{i+1} \frac{\partial^{\ell+k+m+1-i}(L_0 \tilde{\theta})}{\partial t^{\ell-i} \partial \xi^k \partial \xi^{m+i+1}}}, \quad \ell \geq 1.
\]
(163)
Using (66) and (146), we find
\[
\left| \sum_{i=0}^{\ell-1} \varepsilon^{i+1} \frac{\partial^{\ell+k+m+i-1}(L_0 \tilde{\theta})}{\partial t^{\ell-i} \partial \xi^k \partial \xi^{m+i+1}} \right| \leq \kappa_T \varepsilon^{-m-\frac{1}{2}} \int_0^t \{1+(t-s)^{-m-\frac{1}{2}}\} \exp\left(-\frac{\xi_3^2}{8\varepsilon(t-s)}\right) \, ds.
\]
(164)
Hence (152) with \( \ell \geq 1 \) and \( k, m \geq 0 \) follows from (161), (163), and (164), and the lemma is now proved.

Using (139) and Lemmas 2.5 and 2.6, we notice that
\[
\left| \frac{\partial^{\ell+k+m}(\tilde{\theta}^+ - \tilde{\theta}^-)}{\partial t^\ell \partial \xi^k \partial \xi^m} \right| = e.s.t., \quad \ell, k, m \geq 0, \quad d = 0, 1, i = 1, 2,
\]
(165)
for \( (\xi', \xi_3, t) \) in \( \omega_{\varepsilon'} \times (0, \delta) \times (0, T). \)

Now we state and prove the following elementary lemma:

**Lemma 2.7.** For any \( 1 \leq p \leq \infty, \) \( q_1 \geq 0, \) and \( q_2 \geq 1, \) we have, for \( 0 \leq t \leq T, \)
\[
\left\| (\varepsilon^{-\frac{\xi_3}{q_2}} \exp\left(-\frac{\xi_3^2}{4q_2\varepsilon t}\right) \right\|_{L^p(\mathbb{R}_+)} \leq \kappa_T (\varepsilon t)^{\frac{1}{2}}.
\]
(166)
**Proof.** Using the boundedness of the exponentially decaying function, we see that (166) with \( p = \infty \) holds true.

Setting \( \eta = \xi_3/\sqrt{q_2\varepsilon t}, \) we write
\[
\left\| (\varepsilon^{-\frac{\eta^2}{q_2}} \exp\left(-\frac{\eta^2}{4q_2\varepsilon t}\right) \right\|_{L^p(\mathbb{R}_+)}^p = \int_0^\infty (\varepsilon^{-\frac{\eta^2}{q_2}} \xi_3^{q_1} \exp\left(-\frac{p\xi_3^2}{4q_2\varepsilon t}\right)) \, d\xi_3
\]
\[
\leq \kappa_T (\varepsilon t)^{\frac{1}{2}} \int_0^\infty (\eta^{q_1} \exp\left(-\eta^2/4d\eta\right)) \leq \kappa_T (\varepsilon t)^{\frac{1}{2}}.
\]
(167)
Thus (166) follows for $1 \leq p < \infty$. \hfill \square

Thanks to Lemmas 2.5, 2.6, and 2.7, and (165), we find the $L^p$ estimates of $\theta^{d/2}$ and $\theta^{1/2}$, $d = 0, 1$,

\[
\left\| \left( \frac{\xi_3}{\varepsilon} \right)^q \frac{\partial^{k+m} \theta^0}{\partial \xi_i \partial \xi_j} \right\|_{L^p(\omega_c \times \mathbb{R}^+)} \leq \kappa_T(\varepsilon t)^{\frac{1}{2p} - \frac{q}{p}},
\]

for $i = 1, 2$, $1 \leq p < \infty$, $q \geq 0$, $k \geq 0$, and $0 \leq m \leq 3$, and

\[
\left\| \left( \frac{\xi_3}{\varepsilon} \right)^q \frac{\partial^{k+m} \theta^+}{\partial \xi_i \partial \xi_j} \right\|_{L^p(\omega_c \times \mathbb{R}^+)} \leq \kappa_T(\varepsilon t)^{\frac{1}{2p} - \frac{q}{p}},
\]

for $i = 1, 2$, $1 \leq p < \infty$, $q \geq 0$, $k \geq 0$, and $m = 0, 1$.

Using (132), (133), (139), and (165), we write the equations for $\theta^{d/2}$, $d = 0, 1$,

\[
\left\{ \begin{array}{ll}
\frac{\partial \theta^0}{\partial t} - \varepsilon \frac{\partial^2 \theta^0}{\partial \xi_i \partial \xi_j} = e.s.t., & \text{in } \Omega \times (0, T), \\
\theta^0 = -u^0, & \text{on } \Gamma \times (0, T), \\
\theta^0|_{t=0} = 0, & \text{in } \Omega,
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{ll}
\frac{\partial \theta^+}{\partial t} - \varepsilon \frac{\partial^2 \theta^+}{\partial \xi_i \partial \xi_j} = \sqrt{\varepsilon} L_0 \theta^0 + e.s.t., & \text{in } \Omega \times (0, T), \\
\theta^+ = 0, & \text{on } \Gamma \times (0, T), \\
\theta^+|_{t=0} = 0, & \text{in } \Omega.
\end{array} \right.
\]

We introduce the difference between the heat solution $u^\varepsilon$ and the proposed asymptotic expansions at order $\varepsilon^0$ and $\varepsilon^{1/2}$ in the form,

\[
w_{\varepsilon,0} := u^\varepsilon - (u^0 + \theta^0), \quad w_{\varepsilon, \frac{1}{2}} := u^\varepsilon - (u^0 + \theta^0 + \varepsilon^{\frac{1}{2}} \theta^+).
\]

Now we state and prove the validity of the asymptotic expansion (121) as well as the convergence of $u^\varepsilon$ to $u^0$ in the theorem below.

**Theorem 2.4.** Assuming that $u_0$ belongs to $H^2(\Omega)$ and satisfies the compatibility condition (119), and $f$ belongs to $\{ f \in C^1([0,T]; H^2(\Omega)), \quad f|_{\Gamma} \in C^1([0,T]; W^{2,\infty}(\Gamma)) \}$, the difference $w_{\varepsilon,0}$ between the heat solution $u^\varepsilon$ and its asymptotic expansion of order $\varepsilon^{d/2}$, $d = 0, 1$, vanishes as the diffusivity parameter $\varepsilon$ tends to zero in the sense that

\[
\left\| \frac{\partial^m w_{\varepsilon,0}}{\partial \xi_i \partial \xi_j} \right\|_{L^\infty(0,T; L^2(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2} - \frac{m}{2}}, \quad \left\| \frac{\partial^m w_{\varepsilon,0}}{\partial \xi_i \partial \xi_j} \right\|_{L^2(0,T; H^1(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2} - \frac{m}{2}},
\]

for $m = 0, 1$, and that

\[
\left\| w_{\varepsilon, \frac{1}{2}} \right\|_{L^\infty(0,T; L^2(\Omega))} \leq \kappa_T \varepsilon, \quad \left\| w_{\varepsilon, \frac{1}{2}} \right\|_{L^2(0,T; H^1(\Omega))} \leq \kappa_T \varepsilon^{\frac{1}{2}},
\]

for $m = 0, 1$.
for a constant $\kappa_T$ depending on $T$ and the other data, but independent of $\varepsilon$. Moreover, as $\varepsilon$ tends to zero, $u^\varepsilon$ converges to the limit solution $u^0$ in the sense that

$$\|u^\varepsilon - u^0\|_{L^\infty(0,\varepsilon;L^2(\Omega))} \leq \kappa \varepsilon^{1/2}. \quad (175)$$

We also have

$$\lim_{\varepsilon \to 0} \left( \frac{\partial u^\varepsilon}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} = \left( \frac{\partial u^0}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} - \left( u^0, \varphi \right)_{L^2(\Gamma)},$$

for all $\varphi \in C(\bar{\Omega})$, uniformly in time, $t \in [0, T]$, which expresses the fact that

$$\lim_{\varepsilon \to 0} \frac{\partial u^\varepsilon}{\partial \xi_3} = \frac{\partial u^0}{\partial \xi_3} - u^0(\cdot, 0, \cdot) \delta T^3,$$

uniformly in time, $t \in [0, T]$, in the sense of weak convergence of bounded measures on $\Omega$.

Proof. Using (8), (118), (122), (170), and (171), we write the equation for $w^\varepsilon/d/2$, $d = 0, 1$, as

$$\begin{cases}
\frac{\partial w^\varepsilon}{\partial \varepsilon} - \varepsilon \Delta w^\varepsilon = \varepsilon \Delta w^0 + R^\varepsilon + e.s.t., & \text{in } \Omega \times (0, T), \\
w^\varepsilon = 0, & \text{on } \Gamma \times (0, T), \\
w^\varepsilon|_{t=0} = 0, & \text{in } \Omega,
\end{cases} \quad (178)$$

where

$$\begin{cases}
R_0 = \varepsilon S\theta^0 + \varepsilon L\theta^0, \\
R^\varepsilon = \varepsilon S\theta^0 + \varepsilon(L - L_0)\theta^0 + \varepsilon S\theta^\varepsilon + \varepsilon L\theta^\varepsilon + \varepsilon S\theta^\varepsilon + \varepsilon L\theta^\varepsilon.
\end{cases} \quad (179)$$

Using (9), (168), (67), and (169), we notice that

$$\|R_0\|_{L^2(\Omega)} \leq \kappa \varepsilon \left\| \frac{\partial^2 \theta^0}{\partial \xi_i^2} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\partial \theta^0}{\partial \xi_3} \right\|_{L^2(\Omega)} \leq \kappa_T \varepsilon^{1/2}, \quad i = 1, 2, \quad (180)$$

and that

$$\begin{aligned}
\|R^\varepsilon\|_{L^2(\Omega)} &\leq \kappa \varepsilon \left\| \frac{\partial^2 \theta^0}{\partial \xi_i^2} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\xi_3}{\sqrt{\varepsilon}} \frac{\partial \theta^0}{\partial \xi_3} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\partial^2 \theta^\varepsilon}{\partial \xi_i^2} \right\|_{L^2(\Omega)} + \kappa \varepsilon \left\| \frac{\partial \theta^\varepsilon}{\partial \xi_3} \right\|_{L^2(\Omega)} \\
&\leq \kappa_T \varepsilon^{1/2}, \quad i = 1, 2.
\end{aligned} \quad (181)$$

Using these estimates, we multiply (178) by $w^\varepsilon/d/2$, integrate over $\Omega$, and integrate by parts. Applying the Schwarz inequality as well, we find that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left\| w^\varepsilon/d \right\|_{L^2(\Omega)}^2 + \varepsilon \left\| \nabla w^\varepsilon/d \right\|_{L^2(\Omega)}^2 \\
&\leq \left\| w^\varepsilon/d \right\|_{L^2(\Omega)}^2 + \kappa \varepsilon \left\| \Delta u^0 \right\|_{L^2(\Omega)}^2 + \kappa \sum_{n=0}^d \|R^\varepsilon\|_{L^2(\Omega)}^2 \\
&\leq \left\| w^\varepsilon/d \right\|_{L^2(\Omega)}^2 + \kappa_T \varepsilon^{1/2}, \quad d = 0, 1.
\end{aligned} \quad (182)$$

Hence we obtain (173) with $m = 0$ and (174) using the Gronwall lemma.

---

3Here again $\delta T^3$ denotes the delta measure supported on the boundary and is not related to the “small” coefficient $\delta$. 
The convergence result (175) follows from (173) with \( m = 0 \) and (168), thanks to the triangle inequality.

To verify (173) with \( m = 1 \), we differentiate (178) with \( m = 0 \) in the normal variable \( \xi_3 \) and find the equation and initial condition for \( \partial w_{\varepsilon,0}/\partial \xi_3 \),

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right) - \varepsilon \Delta \left( \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right) &= \varepsilon \frac{\partial \Delta u^0}{\partial \xi_3} + \frac{\partial R_0}{\partial \xi_3} + \text{e.s.t.}, \quad \text{in } \Omega \times (0,T), \\
\left. \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right|_{t=0} &= 0, \quad \text{in } \Omega.
\end{align*}
\]  

(183)

Using (8), we restrict (178) with \( m = 0 \) on \( \Gamma, (\xi_3 = 0) \) and find that

\[-\varepsilon L w_{\varepsilon,0} - \varepsilon \frac{\partial^2 w_{\varepsilon,0}}{\partial \xi_3^2} = \varepsilon(\Delta u^0 + L \theta^0), \quad \text{on } \Gamma \times (0,T).\]  

(184)

Here we used the fact that the e.s.t. on the right-hand side of (178) vanishes on \( \Gamma \). Using (8) and (184), we now obtain the Robin boundary condition for \( \partial w_{\varepsilon,0}/\partial \xi_3 \) in the form,

\[
\frac{\partial^2 w_{\varepsilon,0}}{\partial \xi_3^2} + \left( \frac{1}{h} \frac{\partial h}{\partial \xi_3} \right) \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} = -\Delta u^0 - \left( \frac{1}{h} \frac{\partial h}{\partial \xi_3} \right) \frac{\partial \theta^0}{\partial \xi_3}, \quad \text{on } \Gamma \times (0,T). \]  

(185)

We multiply (183) with \( \partial w_{\varepsilon,0}/\partial \xi_3 \), integrate over \( \Omega \), and integrate by parts. After applying the Schwarz inequality, we find that

\[
\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \varepsilon \left\| \nabla \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 \leq \varepsilon \int_{\Gamma} \left( \nabla \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right) \cdot \mathbf{n} \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \, dx \\
+ \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \kappa \varepsilon \left\| \frac{\partial \Delta u^0}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \kappa \left\| \frac{\partial R_0}{\partial \xi_3} \right\|_{L^2(\Omega)}^2. \]  

(186)

Using (185) and the fact that \( e_3 = -\mathbf{n} \) on \( \Gamma \), we estimate the first term on the right-hand side of (186),

\[
\left| \varepsilon \int_{\Gamma} \left( \nabla \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right) \cdot \mathbf{n} \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \, dx \right| \leq \varepsilon \int_{\Gamma} \left| \frac{\partial^2 w_{\varepsilon,0}}{\partial \xi_3^2} \right| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \, dx \\
\leq \kappa \varepsilon \left\| \frac{\partial^2 w_{\varepsilon,0}}{\partial \xi_3^2} \right\|_{L^2(\Gamma)} \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Gamma)} \\
\leq \kappa \varepsilon \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Gamma)}^2 + \kappa \varepsilon \left[ 1 + \left\| \frac{\partial \theta^0}{\partial \xi_3} \right\|_{L^2(\Gamma)} \right] \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Gamma)}. \]  

(187)

Thanks to the trace theorem, we notice that

\[
\kappa \varepsilon \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Gamma)}^2 \leq \kappa \varepsilon \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{L^2(\Omega)} \left\| \frac{\partial w_{\varepsilon,0}}{\partial \xi_3} \right\|_{H^1(\Omega)}.
\]
\[
\leq \kappa \varepsilon \left\| \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \kappa \varepsilon \left\| \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)} \left\| \nabla \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}
\]

\leq \kappa \varepsilon \left\| \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \varepsilon \left\| \nabla \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2.
\]  

Using (146), (165), and the continuity of \( \theta^0 \), we infer that
\[
\left\| \frac{\partial \theta^0}{\partial \xi_3} \right\|_{L^2(\Gamma)} \leq \kappa_T \varepsilon^{-\frac{1}{2}}.
\]

Hence, using the trace theorem as well, we estimate the second term on the right-hand side of (187),
\[
\kappa \varepsilon \left\| \frac{\partial R_0}{\partial \xi_3} \right\|_{L^2(\Omega)} \leq \kappa \varepsilon \left\| \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \varepsilon \left\| \nabla \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2.
\]

We deduce from (187), (188), and (190) that
\[
\varepsilon \int_{T} \left( \nabla \frac{\partial w_{z,0}}{\partial \xi_3} \right) \cdot n \frac{\partial w_{z,0}}{\partial \xi_3} \, dx \leq \kappa \varepsilon \left\| \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \varepsilon \left\| \nabla \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2.
\]

Using (9), (168), and (179), we observe that
\[
\left\| \frac{\partial R_0}{\partial \xi_3} \right\|_{L^2(\Omega)} \leq \kappa \varepsilon \left\| \frac{\partial \theta^0}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \kappa \varepsilon \left\| \frac{\partial^2 \theta^0}{\partial \xi_3^2} \right\|_{L^2(\Omega)}^2 \leq \kappa_T \varepsilon^{-\frac{1}{2}}, \quad i = 1, 2.
\]

Combining (186), (191), and (192), we see that
\[
\frac{d}{dt} \left\| \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 + \varepsilon \left\| \nabla \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2 \leq \kappa_T \varepsilon^{-\frac{1}{2}} + \kappa_T \left\| \frac{\partial w_{z,0}}{\partial \xi_3} \right\|_{L^2(\Omega)}^2,
\]

and (173) follows from (193) for \( m = 1 \).

Now to show the weak convergence result in (176), we infer from (173) with \( m = 1 \) that
\[
\left\| \left( \frac{\partial w}{\partial \xi_3} - \left( \frac{\partial \theta^0}{\partial \xi_3} + \frac{\partial \theta^0}{\partial \xi_3} \right), \varphi \right) \right\|_{L^2(\Omega)} \leq \kappa_T \varepsilon^{-\frac{1}{2}}, \quad \forall \varphi \in C(\overline{\Omega}),
\]

uniformly in \( 0 < t < T \).

Using (139) and (146), we notice that
\[
\left( \frac{\partial \theta^0}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} = \left( \frac{\partial \theta^0}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} + e.s.t.,
\]
Hence we deduce from (194) that
\[
\lim_{\varepsilon \to 0} \left( \frac{\partial u_3^\varepsilon}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} = \left( \frac{\partial u_0}{\partial \xi_3}, \varphi \right)_{L^2(\Omega)} + \lim_{\varepsilon \to 0} \left( \frac{\partial \sigma}{\partial \xi_3} \varphi \right)_{L^2(\Omega)}, \quad \forall \varphi \in C(\overline{\Omega}),
\]
(196)
uniformly in \(0 < t < T\), if the limit on the right-hand side exists.

We observe from (168) that the \(L^\infty(0, T; L^1(\Omega))\) norm of \(\partial \sigma / \partial \xi_3\) is bounded independently of \(\varepsilon\). Therefore the limit of \(\partial \sigma / \partial \xi_3\) at \(\varepsilon = 0\) must exist in the space of Radon measures in \(\overline{\Omega}\) uniformly in time \(0 < t < T\).

To find the explicit form of the limit for \(\partial \sigma / \partial \xi_3\) as \(\varepsilon\) tends to zero, we fix \(0 < t < T\) and introduce the following approximation of the \(\delta\)-measure in \(\mathbb{R}\),
\[
\eta_\varepsilon(x, t) = \frac{\partial}{\partial x} \left\{ \text{erfc} \left( \frac{x}{\sqrt{2 \varepsilon t}} \right) \right\} = \frac{1}{2 \sqrt{\pi \varepsilon t}} e^{-x^2/(4 \varepsilon t)},
\]
(197)
so that \(\|\eta_\varepsilon\|_{L^1(\mathbb{R})} = 1\) for all \(\varepsilon, t > 0\). Then, using (134) and (197), we write
\[
\frac{\partial \sigma}{\partial \xi_3} (\varphi, \xi)_{L^2(\Omega)}
\]
\[
= -2 \int_0^t \int_{\omega_{\varepsilon}} \frac{\partial u_0}{\partial t} (\xi', 0, t - s) \left( \int_0^\infty \eta_\varepsilon (\xi_3, s) \sigma \varphi d\xi_3 \right) d\xi' ds
\]
\[
= - \int_0^t \int_{\omega_{\varepsilon}} \frac{\partial u_0}{\partial t} (\xi', 0, t - s) \left( \int_\mathbb{R} \eta_\varepsilon (\xi_3) \sigma(|\xi_3|) \varphi (\xi', |\xi_3|) h(\xi', |\xi_3|) d\xi_3 \right) d\xi' ds.
\]
(198)
Since \(\eta_\varepsilon\) is an approximation of the \(\delta\)-measure, the most inner integral with respect to \(\xi_3\) converges to \(\sigma \varphi h\) evaluated at \(\xi_3 = 0\) as \(\varepsilon\) tends to zero. Using this fact, (124), and \(\sigma(0) = 1\), we deduce from (198) that
\[
\lim_{\varepsilon \to 0} \frac{\partial \sigma}{\partial \xi_3} (\varphi, \xi)_{L^2(\Omega)} = - \int_{\omega_{\varepsilon}} \int_0^t \frac{\partial u_0}{\partial t} (\xi', 0, t - s) \varphi(\xi', 0) h(\xi', 0) d\xi' ds
\]
\[
= - \int_{\omega_{\varepsilon}} u_0(\xi', 0, t) \varphi(\xi', 0) h(\xi', 0) d\xi'
\]
(199)
\[
= - \left( u_0, \varphi \right)_{L^2(\Gamma)}, \quad \forall \varphi \in C(\overline{\Omega}).
\]
Hence (176) follows from (196) and (199), and now the proof of Theorem 2.4 is complete.

Remark 2.7. The weak convergence result (176) in the space of Radon measures is borrowed from [44].

2.3.2. Boundary layer analysis at arbitrary orders \(\varepsilon^n\) and \(\varepsilon^{n+1/2}\), \(n \geq 0\).

To extend the convergence results of the heat solution \(u^\varepsilon\) of (118) in Theorem 2.4, we construct the asymptotic expansions \(u_0^n\) and \(u_0^{n+1/2}\) of \(u^\varepsilon\) at arbitrary orders \(n\) and \(n + 1/2\), \(n \geq 0\), in the form,
\[
\begin{cases}
\quad u_0^n = \sum_{j=0}^n \left( \varepsilon^j u^j + \varepsilon^{j+1/2} \theta^{j+1/2} \right), \\
\end{cases}
\]
\[
\begin{cases}
\quad u_0^{n+1/2} = \sum_{j=0}^n \left( \varepsilon^j u^j + \varepsilon^{j+1/2} \theta^{j+1/2} \right),
\end{cases}
\]
(200)
where the \( u^j \) correspond to the external expansion and the correctors \( \theta^j \) and \( \theta^{j+1/2} \) correspond to the inner expansion.

To obtain the external expansion of \( u^\varepsilon \), we formally insert the external expansion \( u^\varepsilon \equiv \sum_{j=0}^{\infty} \varepsilon^j u^j \) into (118) and write

\[
\sum_{j=0}^{n} \left( \varepsilon^j \frac{\partial u^j}{\partial t} - \varepsilon^{j+1} \Delta u^j \right) \approx f. \tag{201}
\]

By matching the terms of the same order \( \varepsilon^j \), we write the equation for each \( u^j \),

\[
\frac{\partial u^j}{\partial t} = f, \quad \text{for } j = 0, \text{ and } \Delta u^{j-1}, \quad \text{for } 1 \leq j \leq n. \tag{202}
\]

Each equation is supplemented with the initial condition,

\[
u^j \big|_{t=0} = u_0, \quad \text{for } j = 0, \text{ and } 0, \quad \text{for } 1 \leq j \leq n. \tag{203}
\]

By sequentially solving the initial value problem (202)-(203), we find that

\[
u^j(x,t) = \begin{cases} \text{(right-hand side of (123)),} & j = 0, \\ \frac{1}{j!} \int_{0}^{t} \frac{1}{j!} (t-s)^j \Delta^j f(x,s) \, ds, & 1 \leq j \leq n. \end{cases} \tag{204}
\]

Each \( \nu^j, 0 \leq j \leq n, \) belongs to \( C^{k+j+1}([0,T];H^m(\Omega)) \) for any \( T > 0 \) and \( k,m \geq 0 \), provided that

\[u_0 \in H^{m+2j}(\Omega), \quad f \in C^k([0,T];H^{m+2j}(\Omega)).\]

Note that the \( \nu^j, 0 \leq j \leq n, \) do not necessarily vanish on the boundary \( \Gamma \), and in fact the discrepancy between \( u^\varepsilon \) and \( \sum_{j=0}^{n} \varepsilon^j u^j \) on \( \Gamma \) creates boundary layers near \( \Gamma \).

To balance the discrepancy of \( u^\varepsilon \) and the external expansion, we follow the methodology introduced in Section 2.2.3 and propose an inner expansion near \( \Gamma \) in the form,

\[u^\varepsilon - \sum_{j=0}^{n} \varepsilon^j u^j \cong \sum_{j=0}^{n} \left( \varepsilon^j \theta^j + \varepsilon^{j+\frac{1}{2}} \Delta \theta^{j+\frac{1}{2}} \right), \quad \text{at least near } \Gamma. \tag{205}\]

The main role of \( \theta^j, 0 \leq j \leq n, \) is to balance the discrepancy between \( u^\varepsilon \) and the proposed external expansion at order \( \varepsilon^j \) while \( \theta^{j+1/2}, 0 \leq j \leq n, \) is necessary to be added in the inner expansion to handle the error caused by the curvature of the boundary. As it appears in Theorems 2.5 below, adding the corrector \( \theta^{j+1/2} \) in the expansion ensures the optimal convergence rate at each order of \( \varepsilon^j \), \( 0 \leq j \leq n \).

By matching the terms of the same order \( \varepsilon^j \) on \( \Gamma \), we find the boundary and initial conditions for each \( \theta^j \) from (205):

\[\theta^j = -u^j \quad \text{and} \quad \theta^{j+\frac{1}{2}} = 0, \quad \text{on } \Gamma \times (0,T), \quad 0 \leq j \leq n, \tag{206}\]

and

\[\theta^j \big|_{t=0} = \theta^{j+\frac{1}{2}} \big|_{t=0} = 0, \quad \text{in } \Omega, \quad 0 \leq j \leq n. \tag{207}\]

To find the proper equations for \( \theta^j \) and \( \theta^{j+1/2}, 0 \leq j \leq n, \) we use (118), (204), and (205) to write,

\[
\sum_{j=0}^{\infty} \left( \varepsilon^j \frac{\partial \theta^j}{\partial t} + \varepsilon^{j+\frac{1}{2}} \frac{\partial \theta^{j+\frac{1}{2}}}{\partial t} \right) - \varepsilon \sum_{j=0}^{n} \left( \varepsilon^j \Delta \theta^j + \varepsilon^{j+\frac{1}{2}} \Delta \theta^{j+\frac{1}{2}} \right) \cong 0, \quad \text{at least near } \Gamma. \tag{208}\]
Following the procedure in Section 2.2.3, we use (9), (93), and (94), and collect all terms of order $\varepsilon^j$ in (208) to find the equation of $\theta^j$ and $\theta^{j+1/2}$,

$$
\frac{\partial \theta^j + \frac{1}{2}}{\partial t} - \varepsilon \frac{\partial^2 \theta^j + \frac{1}{2}}{\partial \xi_3^2} = f^j + \frac{1}{2} (\theta), \quad d = 0, 1, \quad \text{at least in } \Omega_3,
$$

(209)

where

$$
f^j + \frac{1}{2} (\theta) := \sum_{k=0}^{2j-2} \varepsilon^{\frac{1}{2}} \xi_3^k S_\varepsilon \theta^{j-1} - \frac{1}{2} + \sum_{k=0}^{2j-1} \varepsilon^{\frac{1}{2}} \xi_3^k L_\varepsilon \theta^{j-\frac{1}{2} - \frac{1}{2}},
$$

and

$$
f^j + \frac{1}{2} (\theta) := \sum_{k=0}^{2j-1} \varepsilon^{\frac{1}{2}} \xi_3^k S_\varepsilon \theta^{j-\frac{1}{2} - \frac{1}{2}} + \sum_{k=0}^{2j} \varepsilon^{\frac{1}{2}} \xi_3^k L_\varepsilon \theta^{j-\frac{1}{2}}.
$$

(210)

The equations above with $j = 0$ are identical to those in (129) and (131).

Modifying the equations (209) and (210), and using the boundary and initial conditions (206) and (207), we define $\overline{\theta}^j$ and $\overline{\theta}^{j+1/2}$, $0 \leq j \leq n$, as solutions of

$$
\begin{align*}
\overline{\theta}^j &= -u^j, \quad \text{at } \xi_3 = 0, \\
\overline{\theta}^j &\rightarrow 0, \quad \text{as } \xi_3 \rightarrow \infty,
\end{align*}
$$

(211)

and

$$
\begin{align*}
\overline{\theta}^{j+\frac{1}{2}} &= 0, \quad \text{at } \xi_3 = 0, \\
\overline{\theta}^{j+\frac{1}{2}} &\rightarrow 0, \quad \text{as } \xi_3 \rightarrow \infty,
\end{align*}
$$

(212)

where

$$
f^j + \frac{1}{2} (\theta) := \text{(the right-hand side of (210) with } \theta \text{ replaced by } \overline{\theta}), \quad d = 0, 1.
$$

(213)

The equations above with $n = 0$ are identical to (132) and (133) with

$$
f^0 + \frac{1}{2} (\theta) = 0, \quad f^j + \frac{1}{2} (\theta) = \sqrt{\varepsilon} L_0 \overline{\theta}^0.
$$

(214)

Thanks to (203) and the linearity of the equation (211), we infer that the solution $\overline{\theta}^j$, $j \geq 1$, is given in the form,

$$
\overline{\theta}^j = \overline{\theta}^j + \overline{\theta}^j_p,
$$

(215)

where

$$
\begin{align*}
\overline{\theta}^j &= (134) \text{ with } u^0 \text{ replaced by } u^j, \\
\overline{\theta}^j_p &= (137) \text{ with } \sqrt{\varepsilon} L_0 \overline{\theta}^0 \text{ replaced by } f^j + \frac{1}{2} (\theta).
\end{align*}
$$

(216)

In the same manner, we find that the solution $\overline{\theta}^{j+1/2}$, $j \geq 1$, of (212) is given in the form,

$$
\overline{\theta}^{j+\frac{1}{2}} = (137) \text{ with } \sqrt{\varepsilon} L_0 \overline{\theta}^0 \text{ replaced by } f^j + \frac{1}{2} (\theta).
$$

(217)

Now, we prove the following pointwise estimates for $\overline{\theta}^j$ and $\overline{\theta}^{j+1/2}$.
Lemma 2.8. Assuming that $u_0$ satisfies the compatibility condition (119) and $f_T$ belongs to $C^1([0,T];W^{2n+k,\infty}(\Gamma))$, the approximate corrector $\hat{\theta}^{i+d/2}$, $0 \leq j \leq n$, $d = 0, 1$, satisfies the pointwise estimates,

$$\left| \frac{\partial^{k+m} \hat{\theta}^{i+d/2}}{\partial \xi_k^l \partial \xi_3^m} \right| \leq \kappa_T \varepsilon^{-\frac{q}{2}} \exp \left( -\frac{\xi_3^2}{2m_4^{2j+d+1}\varepsilon^k} \right), \quad (\xi, t) \in \omega_{\bar{\xi}} \times \mathbb{R}_+ \times (0, T),$$

(218)

for $k \geq 0$ and $m = 0, 1$, and

$$\left| \frac{\partial^{j+k+m} \hat{\theta}^{i+d/2}}{\partial \xi_k^l \partial \xi_3^m} \right| \leq \kappa_T \varepsilon^{-\frac{q}{2}} \int_0^t (1 + s^{2-\ell-m-\frac{1}{2}}) \exp \left( -\frac{\xi_3^2}{4^{j+d+1}\varepsilon^{q}} \right) ds,$$

(219)

for $(\xi, t) \in \omega_{\bar{\xi}} \times (0, \delta) \times (0, T)$, $\ell = 0$, $k \geq 0$, and $m \geq 2$, or $\ell \geq 1$ and $k, m \geq 0$. The constant $\kappa_T$ (or $\kappa_{T,i}$) depends on $T$ (or $T$ and $\delta$) and the other data, but is independent of $\varepsilon$.

Proof. We proceed by induction on $j$.

Thanks to Lemmas 2.5 and 2.6, we immediately see that (218) and (219) hold true when $j$ is equal to 0.

If we assume that (218) and (219) are valid when $j \leq k$ and $d = 0, 1$, then, using (213) with $d = 0$ and the inductive assumption, we notice that

$$\left| \frac{\partial f^{k+1} \hat{\theta}}{\partial \xi_3^l} \right| \leq \kappa_T \left\{ 1 + (\varepsilon^{-\frac{1}{2}}\xi_3)^{2k+1} \right\} \exp \left( -\frac{4^{-2k+2}}{2^{2j+d+1}\varepsilon^k} \right).$$

(220)

Then, using the same arguments as in the proofs of Lemmas 2.5 and 2.6, one can verify that (218) and (219) hold true when $j = k + 1$ and $d = 0$.

Using (218) and (219) for $\hat{\theta}^{k+1}$ and the inductive assumption, we repeat the proof of Lemma 2.6 and verify that (218) and (219) hold true for $\hat{\theta}^{k+3/2}$. By the induction, the proof is now complete. \hfill \Box

Using the cut-off function $\sigma$ in (41) and the approximate correctors $\hat{\theta}^{i+d/2}$ above, we define the actual correctors in the form,

$$\theta^{i+d/2} (\xi', \xi_3, t) := \hat{\theta}^{i+d/2} (\xi', \xi_3, t) \sigma(\xi_3), \quad 0 \leq j \leq n, \quad d = 0, 1,$$

(221)

which are functions well-defined in $\bar{\Omega} \times [0, T]$. Then, using Lemma 2.8, we notice that

$$\left| \frac{\partial^{j+k+m} (\theta^{i+d/2} - \hat{\theta}^{i+d/2})}{\partial \xi_k^l \partial \xi_3^m} \right| = e.s.t., \quad \ell, k, m \geq 0, \quad 0 \leq j \leq n, \quad d = 0, 1, \quad i = 1, 2,$$

(222)

for $(\xi', \xi_3, t)$ in $\omega_{\bar{\xi}} \times (0, \delta) \times (0, T)$.

Thanks to Lemmas 2.7 and 2.8, and (222), we find the $L^p$ estimates of $\hat{\theta}^{i+d/2}$ and $\theta^{i+d/2}$, $0 \leq j \leq n$, $d = 0, 1$,

$$\left\| \left( \frac{\xi_3}{\varepsilon} \right)^q \frac{\partial^{j+k+m} \hat{\theta}^{i+d/2}}{\partial \xi_k^l \partial \xi_3^m} \right\|_{L^p(\omega_{\bar{\xi}} \times \mathbb{R}_+)} \leq \kappa_T \varepsilon^{\frac{q}{2} - \frac{p}{2}},$$

$$\left\| \left( \frac{\xi_3}{\varepsilon} \right)^q \frac{\partial^{j+k+m} \theta^{i+d/2}}{\partial \xi_k^l \partial \xi_3^m} \right\|_{L^p(\bar{\Omega})} \leq \kappa_\varepsilon \varepsilon^{\frac{q}{2} - \frac{p}{2}},$$

(223)

for $i = 1, 2$, $1 \leq p \leq \infty$, $q \geq 0$, $k \geq 0$, and $m = 0, 1$. 

\hfill \Box
Using (211), (212), (221), and (222), we write the equations for \( \theta_j^{d/2}, 0 \leq j \leq n, d = 0, 1, \)
\[
\begin{cases}
\frac{\partial \theta_j}{\partial t} - \frac{\partial^2 \theta_j}{\partial x^2} = f_j(\theta) + \text{e.s.t.}, & \text{in } \Omega \times (0, T), \\
\theta_j = -u_j, & \text{on } \Gamma \times (0, T), \\
\theta_j|_{t=0} = 0, & \text{in } \Omega,
\end{cases}
\tag{224}
\]
and
\[
\begin{cases}
\frac{\partial \theta_j^{1/2}}{\partial t} - \frac{\partial^2 \theta_j^{1/2}}{\partial x^2} = f_j^{1/2}(\theta) + \text{e.s.t.}, & \text{in } \Omega \times (0, T), \\
\theta_j^{1/2} = 0, & \text{on } \Gamma \times (0, T), \\
\theta_j^{1/2}|_{t=0} = 0, & \text{in } \Omega.
\end{cases}
\tag{225}
\]
We introduce the remainders at order \( \varepsilon^n \) and \( \varepsilon^{n+1/2}, n \geq 0, \) in the form,
\[
w_{\varepsilon,n+d/2} := u^\varepsilon - u_{\varepsilon,n+d/2}, \quad d = 0, 1,
\tag{226}
\]
where the asymptotic expansion \( u_{\varepsilon,n+d/2}, d = 0, 1, \) of \( u^\varepsilon \) is given in (200).

Now we state and prove the validity of the asymptotic expansion as a generalization of Theorems 2.4:

**Theorem 2.5.** Assuming that \( u_0 \) belongs to \( H^{2n+2}(\Omega) \) and satisfies the compatibility condition (119), and that \( f \) belongs to \( \{ f \in C^1([0, T]; H^2n+2(\Omega)), f|_{\Gamma} \in C^1([0, T]; W^{2n+2,\infty}(\Gamma)) \} \), the difference \( w_{\varepsilon,n+d/2} \) between the heat solution \( u^\varepsilon \) and its asymptotic expansion of order \( \varepsilon^{n+d/2}, \) \( d = 0, 1, n \geq 0, \) vanishes as the diffusivity parameter \( \varepsilon \) tends to zero in the sense that
\[
\|w_{\varepsilon,n+d/2}\|_{L^\infty(0,T;L^2(\Omega))} \leq \kappa_T \varepsilon^{n+\frac{d}{2}}, \quad \|w_{\varepsilon,n+d/2}\|_{L^2(0,T;H^1(\Omega))} \leq \kappa_T \varepsilon^{n+\frac{d}{2}},
\tag{227}
\]
for a constant \( \kappa_T \) depending on \( T \) and the other data, but independent of \( \varepsilon. \)

**Proof.** Using (118), (202), (203), and (226), we write the equations for \( w_{\varepsilon,n+d/2}, \) \( d = 0, 1, n \geq 0, \)
\[
\begin{cases}
\frac{\partial w_{\varepsilon,n+d/2}}{\partial t} - \varepsilon \Delta w_{\varepsilon,n+d/2} = \varepsilon^{n+1} \Delta u^n + R_{n+d/2}, & \text{in } \Omega \times (0, T), \\
w_{\varepsilon,n+d/2} = 0, & \text{on } \Gamma \times (0, T), \\
w_{\varepsilon,n+d/2}|_{t=0} = 0, & \text{in } \Omega,
\end{cases}
\tag{228}
\]
where
\[
R_{n+d/2} = \sum_{n=0}^{2n+d} (\varepsilon \Delta \theta^n - \theta^n), \quad d = 0, 1.
\tag{229}
\]
We multiply the equation (228) by \( w_{\varepsilon,n+d/2}, \) integrate over \( \Omega, \) and integrate by parts to find
\[
\frac{d}{dt} \left\|w_{\varepsilon,n+d/2}\right\|_{L^2(\Omega)}^2 + 2\varepsilon \left\|\nabla w_{\varepsilon,n+d/2}\right\|_{L^2(\Omega)}^2 
\leq 2 \left\|w_{\varepsilon,n+d/2}\right\|_{L^2(\Omega)}^2 + \varepsilon^{2n+2} \|\Delta u^n\|_{L^2(\Omega)}^2 + \left\|R_{n+d/2}\right\|_{L^2(\Omega)}^2,
\tag{230}
\]

Using the expression of the Laplacian in (8), (9), and (93), and the equations of the correctors in (224) and (225), we notice that

\[
R_n = \sum_{j=0}^{2n-2} \varepsilon^{\frac{j}{2}+1} \left( S_n - \sum_{k=0}^{2n-j-2} \xi_3^k S_{\frac{k}{2}} \right) \theta^\frac{j}{2} + \varepsilon^{n+\frac{j}{2}} S \theta^{n-\frac{j}{2}} + \varepsilon^{n+1} \theta^n
\]

(231)

and

\[
R_{n+\frac{1}{2}} = \sum_{j=0}^{2n-1} \varepsilon^{\frac{j}{2}+1} \left( S_n - \sum_{k=0}^{2n-j-1} \xi_3^k S_{\frac{k}{2}} \right) \theta^\frac{j}{2} + \varepsilon^{n+\frac{j}{2}} S \theta^{n-\frac{j}{2}} + \varepsilon^{n+1} \frac{\theta^n}{\theta^{n+\frac{1}{2}}}
\]

(232)

We recall that \( S_n \) and the \( S_{k/2} \) are tangential differential operators, and that \( L_n \) and the \( L_{k/2} \) are proportional to \( \partial / \partial \xi_3 \). Hence, using (223), we find that

\[
\left\| R_n \right\|_{L^2(\Omega)} \leq \kappa \left\{ \varepsilon^{n+\frac{j}{2}} \left\| \sum_{k=0}^{2n-2} \left( \varepsilon^{-\frac{j}{2}} \xi_3^k \right)^{2n-k-1} S_{n-\frac{j}{2}+\frac{k}{2}} \theta^\frac{k}{2} \right\|_{L^2(\Omega)} + \varepsilon^{n+1} \left\| S \theta^{n-\frac{j}{2}} \right\|_{L^2(\Omega)}
\]

+ \varepsilon^{n+1} \left\| \theta^n \right\|_{L^2(\Omega)}
\]

(233)

and

\[
\left\| R_{n+\frac{1}{2}} \right\|_{L^2(\Omega)} \leq \kappa \left\{ \varepsilon^{n+\frac{j}{2}} \left\| \sum_{k=0}^{2n-1} \left( \varepsilon^{-\frac{j}{2}} \xi_3^k \right)^{2n-k} S_{n-\frac{j}{2}+\frac{k}{2}} \theta^\frac{k}{2} \right\|_{L^2(\Omega)} + \varepsilon^{n+1} \left\| S \theta^n \right\|_{L^2(\Omega)}
\]

+ \varepsilon^{n+\frac{j}{2}} \left\| S \theta^{n+\frac{1}{2}} \right\|_{L^2(\Omega)} + \varepsilon^{n+1} \left\| \sum_{k=0}^{2n} \left( \varepsilon^{-\frac{j}{2}} \xi_3^k \right)^{2n-k+1} L_{n-\frac{j}{2}+\frac{k}{2}} \theta^\frac{k}{2} \right\|_{L^2(\Omega)}
\]

(234)

\[
+ \varepsilon^{n+\frac{j}{2}} \left\| L \theta^{n+\frac{1}{2}} \right\|_{L^2(\Omega)}
\]

for some constant \( \kappa \).

Thanks to the Gronwall inequality, (227) follows from (230), (233), and (234).

2.3.3. Analysis of the initial layer: The case of ill-prepared initial data.

In this section, we study the asymptotic behavior of the heat solution \( u^t \) of (118) when the initial data is ill-prepared as appearing in (120). As we shall see below, in this interesting case, the so-called initial layer is created at \( t = 0 \) in addition to the
(regular) boundary layers that we analyzed in the previous Sections 2.3.1 and 2.3.2. The initial layer can be understood physically as an impulsively started motion at $t = 0$ near the boundary.

To analyze the initial layer, it is required to introduce an additional corrector ($\tilde{\psi} \sigma$ below in (237)) in the asymptotic expansion $u^\varepsilon \cong u^0 + \theta^0$ (and hence in the expansions at all orders). In fact, in this case, a solution of the equation (132) is given in the form,

$$\Theta^0(\xi, t) = \tilde{\psi} + \bar{\theta}^0,$$

where

$$\begin{cases}
\tilde{\psi}(\xi, t) = -2u_0(\xi', 0) \text{erfc}
\left(\frac{\xi_3}{\sqrt{2\varepsilon t}}\right), \\
\bar{\theta}^0(\xi, t) = (134).
\end{cases}$$

Here the initial layer function $\tilde{\psi}$ corresponds to the ill-prepared initial data $u_0$ on $\Gamma$, and $\tilde{\psi} = 0$ if $u_0$ is well-prepared as in (119).

Thanks to Lemma 2.4, we obtain the following pointwise estimates for $\tilde{\psi}$:

**Lemma 2.9.** Assuming that $u_0|_\Gamma$ belongs to $W^{k,\infty}(\Gamma)$, then $\tilde{\psi}$ satisfies the pointwise estimates,

$$\left|\frac{\partial^{k+m}\tilde{\psi}}{\partial \xi_1^k \partial \xi_3^m}\right| \leq \begin{cases}
\kappa_T \exp \left(-\frac{\xi_3^2}{4\varepsilon t}\right), & k \geq 0, m = 0, \\
\kappa(\varepsilon t)^{-m+\frac{k}{2}} \xi_3^{m-1} \exp \left(-\frac{\xi_3^2}{4\varepsilon t}\right), & k \geq 0, m = 1, 2, \\
\kappa \varepsilon^{-m+\frac{k}{2}}(1+t^{-m+\frac{k}{2}})(1+\xi_3^{m-1}) \exp \left(-\frac{\xi_3^2}{4\varepsilon t}\right), & k \geq 0, m \geq 3,
\end{cases}$$

for $(\xi, t) \in \omega_{\xi'} \times \mathbb{R}_+ \times (0, T)$. The constant $\kappa_T$ depends on $T$ and the other data, but is independent of $\varepsilon$.

Using the cut-off function $\sigma$ in (41), we define

$$\Theta^0(\xi, t) := \Theta^0(\xi, t) \sigma(\xi_3) = \tilde{\psi}(\xi, t) \sigma(\xi_3) + \theta^0(\xi, t),$$

with $\theta^0$ as in (139). Then $\Theta^0$ is well-defined in $\bar{\Omega} \times [0, T]$.

Using (165), (236), and the fact that $\psi$ is a solution of the homogeneous heat equation in $\mathbb{R}_+ \times (0, T)$, we observe that

$$\left|\frac{\partial^{\ell+k+m}(\Theta^0 - \Theta^0)}{\partial \xi_1^\ell \partial \xi_3^k \partial \xi_3^m}\right| = \text{e.s.t.}, \quad \ell, k, m \geq 0, \quad d = 0, 1, i = 1, 2,$n

for $(\xi', \xi_3, t) \in \omega_{\xi'} \times (0, \delta) \times (0, T)$.

Using (238) and the fact that $\Theta^0$ is a heat solution, we find the equation of $\Theta^0$,

$$\begin{cases}
\frac{\partial \Theta^0}{\partial t} - \varepsilon \frac{\partial^2 \Theta^0}{\partial \xi_3^2} = \text{e.s.t.}, & \text{in } \Omega \times (0, T), \\
\Theta^0 = -u^0, & \text{on } \Gamma \times (0, T), \\
\Theta^0|_{t=0} = 0, & \text{in } \Omega.
\end{cases}$$

Thanks to Lemmas 2.5, 2.7, and 2.9, and (238), we find the $L^p$ estimates of $\Theta^0$ and $\Theta^0$. 

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Remark 2.8. Comparing (240) with the $L^p$ estimates of $\theta^0$ in (168), we see that handling $\tilde{\psi}$ (and hence $\psi$, $\Theta^0$, and $\Theta^0$) is more problematic than $\tilde{\vartheta}$ and $\vartheta$, because its derivatives in $\xi_3$ or $t$ are more singular at $t = 0$ than the derivatives of $\vartheta$. However the time singularity in (240) is manageable in the analysis of Theorem 2.6 below because it is integrable in time $t$.

Now we set
\[
\bar{u}_{\varepsilon,n+d/2} := u^\varepsilon - \bar{u}_{\varepsilon,n+d/2}, \quad n \geq 0, d = 0, 1,
\]
where the asymptotic expansion $\bar{u}_{\varepsilon,n+d/2}$ of $u^\varepsilon$ is given by
\[
\bar{u}_{\varepsilon,n+d/2} = (\text{right-hand side of } (200)) \text{ with } \vartheta^0 \text{ replaced by } \Theta^0; \text{ see (139) and (237))},
\]
with $n \geq 0, d = 0, 1$. Remembering that the time singularity in (240) is integrable, one can reproduce the convergence results in Theorems 2.4 and 2.5:

Theorem 2.6. Assuming that the initial data $u_0$ is ill-prepared as appearing in (120), the convergence results in Theorems 2.4 and 2.5 hold true with $w_{\varepsilon,n+d/2}$ replaced by $\bar{u}_{\varepsilon,n+d/2}, n \geq 0, d = 0, 1$, provided that the same regularity assumptions are imposed to the data.

3. Convection-diffusion equations in a bounded interval with a turning point. In this section we consider a one-dimensional singularly perturbed problem which has a single turning point on $\Omega = (-1, 1)$,
\[
\begin{cases}
-\varepsilon^2 \frac{d^2 u^\varepsilon}{dx^2} - b \frac{du^\varepsilon}{dx} = f \text{ in } \Omega, \\
u^\varepsilon(-1) = u^\varepsilon(1) = 0,
\end{cases}
\]
where $0 < \varepsilon << 1, b = b(x), f = f(x)$ are smooth on $[-1, 1]$, and
\[b < 0 \text{ for } x < 0, b = 0 \text{ for } x = 0, b > 0 \text{ for } x > 0,
\]
\[
db \geq c_0 > 0, \forall x \in [-1, 1].
\]

We start with the formal outer expansions $u^\varepsilon \equiv \sum_{j=0}^{\infty} \varepsilon^j u^j_1$ in $x < 0$ and $u^\varepsilon \equiv \sum_{j=0}^{\infty} \varepsilon^j u^j_2$ in $x > 0$. Substituting these expansions in Eq. (243) we find that, by identification at each power of $\varepsilon$,
\[
O(1): -b \frac{d u^0_1}{dx} = f \text{ in } [-1, 0], -b \frac{d u^0_2}{dx} = f \text{ in } [0, 1],
\]
\[
O(\varepsilon): -b \frac{d u^1_1}{dx} = 0 \text{ in } [-1, 0], -b \frac{d u^1_2}{dx} = 0 \text{ in } [0, 1],
\]
\[
O(\varepsilon^j): -b \frac{d u^j_1}{dx} = \frac{d^2 u^{j-2}_1}{dx^2}, \text{ in } [-1, 0], -b \frac{d u^j_2}{dx} = \frac{d^2 u^{j-2}_2}{dx^2}, \text{ in } [0, 1], \text{ for } j \geq 2.
\]
We firstly construct the outer expansions \( u^0_l, u^j_r \) as in (245). Here we impose the inflow boundary conditions,

\[
u^j_l(-1) = u^j_r(1) = 0, \quad j \geq 0,
\]

which will be justified below. We then notice with (245) that \( u^j_l = u^j_r = 0 \) for all odd \( j \)'s, \( 1 \leq j \leq n \). Furthermore, we are able to obtain the following explicit expressions:

\[
u^0_l = - \int_{-1}^{x} b(s)^{-1} f(s) \, ds, \quad u^0_r = \int_{x}^{1} b(s)^{-1} f(s) \, ds,
\]

and for all \( j = 2k, k \geq 1, \)

\[
u^{2k}_l = - \int_{-1}^{x} b(s)^{-1} \frac{d b^{2(k-1)}}{d x^2} (s) \, ds, \quad u^{2k}_r = \int_{x}^{1} b(s)^{-1} \frac{d b^{2(k-1)}}{d x^2} (s) \, ds.
\]

We follow the asymptotic analysis given in [73].

3.1. Interior layer analysis at arbitrary order \( \varepsilon^n \). For later use, we first consider the homogeneous boundary value problem with \( f = 0 \), with non-zero boundary conditions,

\[
\begin{align*}
-\varepsilon^2 \frac{d^2 u^\varepsilon}{d x^2} - b \frac{d u^\varepsilon}{d x} &= 0 \quad \text{in } \Omega, \\
u^\varepsilon(-1) &= \alpha, \quad u^\varepsilon(1) = \beta.
\end{align*}
\]

To enforce the boundary values \( \alpha, \beta \) at \( x = \pm 1 \) we incorporate the so-called interior layer \( \theta^j \).

**Interior layers \( \theta^j \)**

To resolve the discrepancies between \( \alpha \) and \( \beta \) if these numbers are different, we introduce as follows the so-called ordinary interior layers which are defined by the inner expansions \( u^\varepsilon \equiv \sum_{j=0}^{\infty} \varepsilon^j \theta^j \) with a stretched variable \( \bar{x} = x/\varepsilon, \, \theta^j = \theta^j(\bar{x}), \, \bar{x} \in (-\infty, \infty) \). Using the formal Taylor expansion for \( b = b(x) \) at \( x = 0 \) we obtain the asymptotic expansion for \( b \):

\[
b(x) = \sum_{j=1}^{\infty} b_j x^j = \sum_{j=1}^{\infty} b_j \varepsilon^j \bar{x}^j; \tag{249}
\]

note that \( b_0 = b(0) = 0 \) and \( b_1 = b_x(0) \geq 1 \) by (244). Substituting (249) and the inner expansions \( (\sum_{j=0}^{\infty} \varepsilon^j \theta^j) \) for \( b \) and \( u^\varepsilon \), respectively, in Eq. (243), we then obtain (with \( b_0 = 0 \)) the following formal expansion:

\[
\sum_{j=0}^{\infty} \left\{ -\varepsilon^j \frac{d^2 \theta^0}{d \bar{x}^2} - \varepsilon^j \frac{\sum_{k=0}^{j} b_{j-k+1} \varepsilon^j \bar{x}^{j-k+1} \frac{d \theta^0}{d \bar{x}}}{\bar{x}} \right\} = 0. \tag{250}
\]

By identification at each power of \( \varepsilon \), we find

\[
\begin{align*}
O(1) : & -\frac{d^2 \theta^0}{d \bar{x}^2} - b_1 \frac{d \theta^0}{d \bar{x}} = 0, \tag{251a} \\
O(\varepsilon) : & -\frac{d^2 \theta^1}{d \bar{x}^2} - b_1 \frac{d \theta^1}{d \bar{x}} = b_2 \bar{x} \frac{d \theta^0}{d \bar{x}}, \tag{251b} \\
O(\varepsilon^j) : & -\frac{d^2 \theta^j}{d \bar{x}^2} - b_1 \frac{d \theta^j}{d \bar{x}} = \sum_{k=0}^{j-1} b_{j-k+1} \varepsilon^j \bar{x}^{j-k+1} \frac{d \theta^k}{d \bar{x}}. \tag{251c}
\end{align*}
\]
We impose the boundary conditions,
\[ \theta^0(x = -1) = \alpha, \ \theta^0(x = 1) = \beta, \ \text{and} \ \theta^j(x = -1) = \theta^j(x = 1) = 0, \ 1 \leq j \leq n. \] (252)

But for the purpose of the analysis below it is convenient to consider the approximate form of \( \tilde{\theta}^j \), namely \( \tilde{\theta}^j \) satisfying equations (251) on all of \( \mathbb{R} \) (for the variable \( \bar{x} \)) with the following boundary conditions:
\[ \tilde{\theta}^0 \to \alpha \text{ as } \bar{x} \to -\infty, \ \tilde{\theta}^0 \to \beta \text{ as } \bar{x} \to \infty, \]
\[ \tilde{\theta}^j \to 0 \text{ as } \bar{x} \to \pm \infty, \ j \geq 1. \] (253a)

We show that \( \theta^j \) and \( \tilde{\theta}^j \) differ by an exponentially small term (see [73]), and
\[ \tilde{\theta}^0 = c_0^{-1} \left[ \alpha \int_{\bar{x}}^{\infty} \exp \left( -\frac{b_1 s^2}{2} \right) ds + \beta \int_{-\infty}^{\bar{x}} \exp \left( -\frac{b_1 s^2}{2} \right) ds \right], \] (254a)
\[ \tilde{\theta}^1 = (\alpha - \beta)b_2 3^{-1} c_0^{-1} \int_{-\infty}^{\bar{x}} s^3 \exp \left( -\frac{b_1 s^2}{2} \right) ds, \] (254b)
where \( c_0 = \int_{-\infty}^{\infty} \exp(-b_1 s^2/2) ds = \sqrt{2\pi/b_1} \). The following pointwise and norm estimates for \( \theta^j, j \geq 0 \), can then be derived ([73]).

**Lemma 3.1.** There exist positive constants \( \kappa_{jm} \) and \( c \) such that the following pointwise estimates hold
\[ \left| \frac{d^m \theta^j}{dx^m} \right| \leq \kappa_{jm} \left\{ \begin{array}{ll} 1 & \text{for } j = 0 \text{ and } m = 0, \\
\epsilon^{-m} \exp \left( -c \frac{|x|}{\epsilon} \right) & \text{for } j \geq 1 \text{ or } m \geq 1, \end{array} \right. \] (255)
and for \( m \geq 0 \),
\[ \| \theta^j \|_{H^m(-1,1)} \leq \kappa_{jm} \left( 1 + \epsilon^{-m+\frac{1}{2}} \right). \] (256)

We consider the compatible case which we now present.

3.1.1. \( f, b \) compatible. In this section, we consider the problem (243) with \( f \) arbitrary satisfying the compatibility conditions (257) below. If \( f(0) \neq 0 \), since \( b(0) = 0 \), the limit problem \(-bdv^b/dx = f\) has an inconsistency at \( x = 0 \). That is its solution cannot be smooth. To avoid the inconsistency between \( b \) and \( f \), we assume in this section the following compatibility conditions:
\[ \frac{d^i f}{dx^i}(0) = 0, \ i = 0, 1, \cdots, 2n + 1. \] (257)

If (257) does not hold, we will see that the solution \( u^e \) of (243) possesses logarithmic singularities at \( x = 0 \) as already indicated in the Introduction. Thanks to the compatibility conditions (257), recalling the condition for \( b(x) \) as in (244b) the values of \( u^j_l(0^-) \) and \( u^j_l(0^+) \), of \( du^j_l/dx(0^-) \) and \( du^j_l/dx(0^+) \) or of higher order derivatives, are finite if we take \( n \) sufficiently large. More precisely, we have (see [73] for details):

**Lemma 3.2.** Let \( m \geq 1 \) and \( k \geq 0 \). Assume that the compatibility conditions (257) hold. Then, for \( m, k \) such that \( 0 \leq m + 2k \leq 2n + 2 \), there exists a positive constant \( \kappa_{km} \) such that
\[ \left| \frac{d^m u^{2k}}{dx^m}(0^-) \right|, \ \left| \frac{d^m u^{2k}}{dx^m}(0^+) \right| \leq \kappa_{km}. \] (258)
Interior layers $\theta^j, \psi^j, \zeta^j$

Assuming enough compatibility conditions, we can guarantee, as in Lemma 3.2, that
\[ |u_j^i(0^-)|, |u_j^i(0^+)|, ||du_j^i/dx(0^-)||, ||du_j^i/dx(0^+)|| \leq \kappa. \]
In general, $u_j^i(0^-) \neq u_j^i(0^+)$. To resolve these discrepancies at $x = 0$, using the stretched variable $\bar{x} = x/\varepsilon$, we introduce the functions $\theta^j(\bar{x})$, and $\psi^j(\bar{x})$ which are defined as the solutions of
the same equations (251) respectively on $(-\infty, 0)$, and $(0, \infty)$ with the following boundary conditions.

\[
\begin{align*}
\theta^j(\bar{x}) &= u_j^i(0^-), & d\theta^j/dx &= \frac{\partial \theta^j}{\partial x} = \frac{\partial u_j^i}{\partial x}(0^-) \text{ at } \bar{x} = 0, \quad (259a) \\
\theta^j(\bar{x}) &= u_j^i(0^+), & d\theta^j/dx &= \frac{\partial \theta^j}{\partial x} = \frac{\partial u_j^i}{\partial x}(0^+) \text{ at } \bar{x} = 0, \quad (259b)
\end{align*}
\]
which allows us to determine the $\theta^j$, $\psi^j$ explicitly. Notice that $u_j^i = u_j^i = 0$ for $j$ odd. In particular, for $j = 0, 1$, we find

\[
\begin{align*}
\theta^0 &= \frac{du_0^0}{dx}(0^-) \int_{0}^{\bar{x}} \exp \left( -\frac{b_1s^2}{2} \right) ds + u_0^0(0^-), \\
\theta^1 &= -\frac{du_1^0}{dx}(0^-) b_2 3^{-1} \int_{0}^{\bar{x}} s^3 \exp \left( -\frac{b_1s^2}{2} \right) ds.
\end{align*}
\]
Here we note that as $\bar{x} \to \infty$,

\[
\begin{align*}
\theta^0 &\to c \frac{du_0^0}{dx}(0^-) c_{r,0} + u_0^0(0^-) =: c_{r,\infty}^0, \\
\theta^1 &\to -c \frac{du_1^0}{dx}(0^-) b_2 3^{-1} c_{r,1} =: c_{r,\infty}^1,
\end{align*}
\]
where
\[
c_{r,0} = \int_{0}^{\infty} \exp \left( -\frac{b_1s^2}{2} \right) ds, \quad c_{r,1} = \int_{0}^{\infty} s^3 \exp \left( -\frac{b_1s^2}{2} \right) ds.
\]
We denote by $\varphi \cup \psi$ the function on $(-1, 1)$ equal to (the restriction of) $\varphi$ on $(-1, 0)$ and to (the restriction of) $\psi$ on $(0, 1)$ and consider the functions $u_j^i \cup \theta^j$ and $\varphi \cup \psi$. Note that due to (259), these functions belong to $C^1([-1, 1])$ and to $H^2(-1, 1)$. In general, the following pointwise and norm estimates can be derived.

**Lemma 3.3.** Assume that the compatibility conditions (257) hold. Then there exist positive constants $\kappa$ and $c$ such that for $x \in [0, 1]$, $0 \leq j \leq 2n + 2$,

\[
\begin{align*}
\left| \frac{d^m \theta^j}{dx^m} \right|, \left| \frac{d^m \psi^j}{dx^m} \right| &\leq \kappa \left\{ \begin{array}{ll}
1 & \text{for } m = 0, \\
\epsilon^{-m+1} \exp \left( -\frac{x}{\epsilon} \right) & \text{for } m \geq 1,
\end{array} \right. \\
\| \theta^j \|_{H^m(0,1)} + \| \psi^j \|_{H^m(0,1)} &\leq \kappa \left( 1 + \epsilon^{-m+\frac{1}{2}} \right).
\end{align*}
\]

By our construction, we then notice that the functions $g^j := -(u_j^i \cup \theta^j) - (\varphi \cup \psi)$ attain the values $-\theta^j = -c_{r,\infty}^j(\epsilon) + e.s.t$ at $x = -1$ and $-\theta^j = -c_{r,\infty}^j(\epsilon) + e.s.t$ at $x = 1$. To remedy these discrepancies between $g^j$ and $u^j$ at the boundaries $x = -1, 1$ (we recall that $u^j(-1) = u^j(1) = 0$), we introduce interior layers $\zeta^j$ similar to $\theta^j$ but we use different boundary conditions: the $\zeta^j = \zeta^j(\bar{x})$ satisfy (251) and

\[
\zeta^j = -\theta^j, \text{ at } x = -1, \quad \zeta^j = -\theta^j, \text{ at } x = 1 \text{ for } j \geq 0.
\]

The same estimates as in Lemma 3.1 hold for $\zeta^j$ with $j = 0$ and $m = 0$, and $j \geq 1$ or $m \geq 1$ being replaced respectively by $m = 0$, and $m \geq 1$. 

We let
\[ w_{en} = u^e - \xi_{en} - \eta_{en} - \zeta_{en}, \] (265)
where
\[ \xi_{en} = \sum_{j=0}^{2n} \epsilon^j (u_j^1 \cup \theta_j^1), \quad \eta_{en} = \sum_{j=0}^{2n} \epsilon^j (\theta_j^1 \cup u_j^1), \quad \zeta_{en} = \sum_{j=0}^{2n} \epsilon^j \zeta_j. \] (266)

From the outer expansions (245) and the interior layers \( \theta_j^1, \theta_j^1, \zeta_j \), after some elementary calculations, we find that
\[ L_n w_{en} = R_n^1 + R_n^2 + R_n^3 + \text{e.s.t. in } \Omega, \] (267a)
\[ w_{en}(-1) = w_{en}(1) = 0, \] (267b)
where
\[ R_n^1 = \epsilon^{2n+2} \left( \frac{d^2 u_{2n}^1}{dx^2} \cup \frac{d^2 u_{2n}}{dx^2} \right), \] (267c)
\[ R_n^2 = \sum_{j=0}^{2n} \epsilon^j \left( \frac{d\theta_j^1}{dx} \cup \frac{d\theta_j}{dx} \right) R_j^{1,2n}(b), \] (267d)
\[ R_n^3 = \sum_{j=0}^{2n} \epsilon^j \frac{d\zeta_j}{dx} R_j^{1,2n}(b), \] (267e)
with
\[ R_j^{1,2n}(b) = b(x) - \sum_{k=1}^{2n+1-j} b_k x^k. \] (267f)

Here we used the fact that, by permuting the summations:
\[ \sum_{j=0}^{2n} \left\{ \epsilon^j \sum_{k=0}^{j} b_{j-k+1} x^{j-k+1} \frac{d\theta^k}{dx} \right\} = \sum_{j=0}^{2n} \sum_{k=0}^{j} b_{j-k+1} x^{j-k+1} \epsilon_k \frac{d\theta^k}{dx} \]
\[ = \sum_{j=0}^{2n} \frac{d\theta^j}{dx} \left( \sum_{k=1}^{2n+1-j} b_k x^k \right) = \sum_{j=0}^{2n} \frac{d\theta^j}{dx} (b(x) - R_j^{1,2n}(b)). \] (268)

We now estimate the \( L^2 \)- norm of \( R_n^1 \). We first notice that, by Taylor’s expansion,
\[ |R_j^{1,2n}(b)| = \left| b(x) - \sum_{k=1}^{2n+1-j} b_k x^k \right| \leq \kappa |x|^{2n+2-j} \leq \kappa \epsilon^{2n+2-j} |x|^{2n+2-j}. \] (269)

We then estimate the \( L^2 \)- norms of \( R_n^1, R_n^2 \) and \( R_n^3 \). From (258) with \( u_j^1 = u_j^2 = 0 \), \( j \) odd, we note that \( |d^2 u_j^1/dx^2(0^-)|, |d^2 u_j^1/dx^2(0^+)| \leq \kappa_n, \) \( 0 \leq j \leq 2n + 1 \). Hence, we easily find that
\[ \| R_n^1 \|_{L^2(\Omega)} \leq \kappa_n \epsilon^{2n+2}. \] (270)

Using (269) and the pointwise estimates (262), we find
\[ |R_n^2| \leq \kappa_n \epsilon^{2n+2} \sum_{j=0}^{2n} |x|^{2n+2-j} \left( \left| \frac{d\theta^j}{dx} \chi_{0,1} \right| + \left| \frac{d\theta^j}{dx} \chi_{-1,0} \right| \right) \]
\[ \leq \kappa_n \epsilon^{2n+2} \exp \left( -\frac{c|x|}{2\epsilon} \right), \] (271)
where \( \chi_A(x) \) is the characteristic function of the set \( A \), and

\[
\| R_n^2 \|_{L^2(\Omega)} \leq \kappa \varepsilon^{2n+\frac{3}{2}}, \quad \| R_n^3 \|_{L^2(\Omega)} \leq \kappa \varepsilon^{2n+\frac{3}{2}}.
\]

(272)

We define the weighted energy norm:

\[
\| u \|_{\varepsilon} = \| u \|_{L^2(\Omega)} + \sqrt{\varepsilon} \| u \|_{H^1(\Omega)},
\]

(273)

and conclude that we have proved the following:

**Theorem 3.1.** Assume that the compatibility conditions (257) hold. Let \( u^\varepsilon \) be the solution of (243). Then there exists a constant \( \kappa > 0 \) independent of \( \varepsilon \) such that

\[
\| u^\varepsilon - \xi_{en} - \eta_{en} - \zeta_{en} \|_{\varepsilon} \leq \kappa \varepsilon^{2n+\frac{3}{2}},
\]

(274)

\[
\| u^\varepsilon - \xi_{en} - \eta_{en} - \zeta_{en} \|_{H^2(\Omega)} \leq \kappa \varepsilon^{2n-\frac{3}{2}},
\]

(275)

where \( \xi_{en}, \eta_{en} \) and \( \zeta_{en} \) are as in (266).

We now consider the case where \( f \) does not satisfy the compatibility conditions (257).

3.1.2. \( f, b \) noncompatible. We now want to remove the compatibility conditions. For that purpose, we decompose \( f \) as explained below:

\[
f = \hat{f} + \sum_{k=0}^{2n+1} \gamma_k B_k(x),
\]

(276a)

where

\[
B_0 = b_x(x), \quad B_1 = b(x),
\]

(276b)

\[
B_{k+2} = b(x) \int_0^x B_k(s) ds, \quad k \geq 0.
\]

(276c)

Note that since \( d^i B_k / dx^i(0) = 0 \) for \( i < k \) and \( d^k B_k / dx^k(0) \neq 0 \) for \( i = k \) (recall \( b(0) = 0 \) and \( b_x(0) \geq c_0 > 0 \)), we can recursively find all the \( \gamma_k, k \geq 0 \) so that the compatibility conditions (257) for \( f = \hat{f} \) holds for \( 0 \leq i \leq 2n+1 \). For \( f = B_k(x) \), \( k \) odd, it turns out that the outer solutions are bounded in the neighborhood of \( x = 0 \) (see [73]) and this enables us to perform the same asymptotic analysis as before.

Hence, Theorem 3.1 holds for \( f = \hat{f} \), where

\[
\hat{f} = f - \sum_{k=0}^{n} \gamma_{2k} B_{2k}(x).
\]

(277)

Thus, thanks to the superposition of solutions, it suffices to consider the special cases for \( f = B_{2J} \):

\[
\begin{align*}
-\varepsilon^2 \frac{d^2 u^\varepsilon}{dx^2} - b \frac{du^\varepsilon}{dx} &= B_{2J} \quad \text{in } \Omega, \\
u^\varepsilon(-1) = u^\varepsilon(1) &= 0.
\end{align*}
\]

(278)

Following the analysis for \( f = B_{2J} \) in [79] we need the following correctors,
defined as follows:

\[ u_{\ell,m}^* = \sum_{j=0}^{m} \varepsilon^j (u_{l}^{2j}(x) \cup u_{l}^{2j}(0^-)), \quad (279a) \]

\[ u_{rr,m}^* = \sum_{j=0}^{m} \varepsilon^j (u_{r}^{2j}(0^+) \cup u_{r}^{2j}(x)), \quad (279b) \]

\[ \zeta_{c,m}^* = \sum_{j=0}^{m} \varepsilon^j \zeta_{c,j}, \quad (279c) \]

\[ \varphi_{c,p}^* = \sum_{j=2j}^{p} \delta^j \varphi^j, \text{ for } p \geq 2J, \quad (279d) \]

\[ \xi_{c,q}^* = \sum_{j=0}^{q} \varepsilon^j \xi^j, \text{ for } p \geq 2J, \quad (279e) \]

with a parameter \( \delta > 0 \), to be determined below. Here, the functions in (279) are defined as follows:

- For \( u_{l}^{2j}, u_{r}^{2j} \), for \( j' \geq 0 \), \( u_{l}^{2j-1} u_{r}^{2j+1} = 0 \),

\[ u_{l}^{0} = - \int_{-1}^{x} b(s)^{-1} \rho(s) B_{2J}(s) ds, \quad u_{r}^{0} = \int_{x}^{1} b(s)^{-1} \rho(s) B_{2J}(s) ds, \quad (280) \]

where

\[ \rho(x) = \begin{cases} \psi_1 \left( \frac{x}{\varepsilon} \right) & \text{if } |x| < \delta, \\ 1 & \text{if } \delta \leq |x| < \infty, \end{cases} \]

(281)

and for \( j' \geq 1 \),

\[ u_{l}^{2j} = - \int_{-1}^{x} b(s)^{-1} \frac{\partial^2 u_{l}^{2j-1}}{\partial x^2}(s) ds, \quad u_{r}^{2j} = \int_{x}^{1} b(s)^{-1} \frac{\partial^2 u_{r}^{2j-1}}{\partial x^2}(s) ds. \quad (283) \]

- For \( \zeta^{*j} \), using the stretched variable \( \bar{x} = x / \varepsilon \) we write

\[ b(x) = \sum_{j=1}^{\infty} b_{j} x^{j} = \sum_{j=1}^{\infty} b_{j} \varepsilon^{j} \bar{x}^{j}, \quad (284) \]

where \( b_{j} = b^{(j)}(0) / j! \) and we obtain the interior layer equations,

\[ - \frac{d^2 \zeta^{*0}}{dx^2} - b_{1} \bar{x} \frac{d\zeta^{*0}}{dx} = 0, \quad (285a) \]

supplemented with the boundary conditions, \( \zeta^{*j} \to - u_{l}^{2j}(0^+) \) as \( \bar{x} \to -\infty \), \( \zeta^{*j} \to - u_{r}^{2j}(0^-) \) as \( \bar{x} \to \infty \).

- For \( \varphi^j \), using the stretched variable \( \tilde{x} = x / \delta \) we write

\[ b(x) = \sum_{j=1}^{\infty} \tilde{b}_{j} \tilde{x}^{j} = \sum_{j=1}^{\infty} \tilde{b}_{j} \delta^{j} \tilde{x}^{j}, \quad (286) \]
and we obtain an outer expansion,
\[ -\varepsilon^2\delta^2 - \frac{d^2 \varphi_{2J}}{dx^2} - b_1 \frac{d \varphi_{2J}}{dx} = (1 - \rho(\delta x))\delta^{-2J}B_{2J}(\delta x), \]  
\[ -\varepsilon^2\delta^2 - \frac{d^2 \varphi_j}{dx^2} - b_1 \frac{d \varphi_j}{dx} = \sum_{k=2J}^{j-1} b_{j+1-k} \bar{x}^{j+1-k} \frac{d \varphi_k}{dx}, \quad j \geq 2J + 1, \]
 supplemented with the boundary conditions, \( \varphi_j(0) = 0 \), and \( \varphi_j(\pm \infty) = 0 \) as \( \bar{x} \to \pm \infty \).

- For \( \zeta^j \), using the stretched variable \( \bar{x} = x/\varepsilon \), we obtain the interior layer equations,
\[ -\frac{d^2 \zeta^0}{dx^2} - b_1 \frac{d \zeta^0}{dx} = 0, \]  
\[ -\frac{d^2 \zeta^j}{dx^2} - b_1 \frac{d \zeta^j}{dx} = \sum_{k=0}^{j-1} b_{j-k+1} \bar{x}^{j-k+1} \frac{d \zeta_k}{dx}, \quad j \geq 1, \]
 supplemented with the boundary conditions, \( \zeta^0 \to -\varphi^*_\varepsilon \phi(-1) \) as \( \bar{x} \to \infty \), \( \zeta^0 \to -\varphi^*_\varepsilon \phi(1) \) as \( \bar{x} \to -\infty \), \( \zeta^j \to 0 \) as \( \bar{x} \to \pm \infty \), \( j \geq 1 \).

Along with the analysis presented above for \( f = \hat{f} \) as in (277) we conclude for \( f \) noncompatible:

**Theorem 3.2.** Let \( u^\varepsilon \) be the solution of (243). For \( n, N > 0 \) integers, define
\[ \xi_{en,N} = \sum_{j=0}^{2n} \varepsilon^j (\hat{u}^j \cup \hat{\theta}^j) + \sum_{k=0}^{n} \gamma_{2k} \xi_{2k,0,N}, \]  
\[ \eta_{en,N} = \sum_{j=0}^{2n} \varepsilon^j (\hat{\theta}^j \cup \hat{u}^j) + \sum_{k=0}^{n} \gamma_{2k} \eta_{2k,0,N}, \]  
\[ \zeta_{en,N} = \sum_{j=0}^{2n} \varepsilon^j \hat{\zeta}^j + \sum_{k=0}^{n} \gamma_{2k} \zeta_{2k,0,N}, \]  
\[ \varphi_{en,N} = \sum_{k=0}^{\min(N,n)} \gamma_{2k} \varphi_{2k,0,N}, \]
\[ \zeta_{en,N} = \sum_{k=0}^{\min(N,n)} \gamma_{2k} \zeta_{2k,0,N} - k, \]
where the \( \gamma_k \)'s are determined above, \( \hat{u}^j = u^j \), \( \hat{\theta}^j = \theta^j \), \( \hat{\theta}^j = \theta^j \), and \( \hat{\zeta}^j = \zeta_j \) are as in (266) corresponding to \( f = \hat{f} \) as in (277), and \( \xi_{2k,0,N} = \xi^*_{e,6N}, \eta_{2k,0,N} = \eta^*_{e,6N}, \zeta^{2k}_{2k,0,N} = \zeta^*_{e,6N}, \varphi^{2k}_{2k,0,N} = \varphi^*_{e,2N}, \zeta_{2k,0,N} = \zeta^*_{e,N-k}, \) \( \zeta_{2k,0,N-k} = \zeta^*_{e,N-k} \) are as in (279) with \( J = k \) and \( \delta = \varepsilon^\delta \).

Then there exists \( \kappa = \kappa_{n,N} > 0 \), independent of \( \varepsilon \), such that
\[ \| u^\varepsilon - \xi_{en,N} - \eta_{en,N} - \zeta_{en,N} - \varphi_{e,2N} - \zeta_{en,N} \| \leq \kappa (\varepsilon^{2n+\frac{2}{3}} + \varepsilon^{n+\frac{2}{3}}), \]
\[ \| u^\varepsilon - \xi_{en,N} - \eta_{en,N} - \zeta_{en,N} - \varphi_{e,2N} - \zeta_{en,N} \|_{H^2(\Omega)} \leq \kappa \varepsilon^{-3} (\varepsilon^{2n+\frac{2}{3}} + \varepsilon^{n+\frac{2}{3}}). \]

We consider a singularly perturbed convection-diffusion equation in a rectangular domain \( \Omega = (0, 1) \times (0, 1) \):

\[
\begin{align*}
-\varepsilon \Delta u^\varepsilon - \frac{\partial u^\varepsilon}{\partial x} &= f \quad \text{in } \Omega, \\
\quad u^\varepsilon &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]  

(296)

Here \( \varepsilon \) is a small but strictly positive diffusivity parameter, and \( f = f(x, y) \) is a given smooth function with \( \| D^\alpha f \|_{L^2(\Omega)} \leq \kappa_\alpha \), independent of \( \varepsilon \), for some \( \alpha \)'s as needed in the analysis below.

In a related earlier work, [122], the following problem similar to (296) was discussed. More precisely, in a rectangular domain \( \tilde{\Omega} = (0, l_0) \times (0, l_1) \), the authors considered:

\[
\begin{align*}
-\varepsilon' \Delta u^{\varepsilon'} - b \frac{\partial u^{\varepsilon'}}{\partial x} + cu^{\varepsilon'} &= \tilde{f} \quad \text{in } \tilde{\Omega}, \\
\quad u^{\varepsilon'} &= \tilde{g} \quad \text{on } \partial \tilde{\Omega}.
\end{align*}
\]  

(297)

where \( b > 0, c \geq 0 \) are constants, \( \tilde{f} \) is a given smooth function in \( \tilde{\Omega} \). The function \( \tilde{g} = \tilde{g}(x, y) \) is assumed to be continuous on \( \partial \tilde{\Omega} \) and smooth on each edge of \( \partial \tilde{\Omega} \).

Using a simple change of variables which maps \( \tilde{\Omega} \) onto \( \Omega \), and setting \( u^{\varepsilon'} = u^\varepsilon e^{\lambda x} \) with a suitable \( \lambda \), our analysis of (296) in this article is applicable to (297) as well (see [42]).

We will use a smooth cut-off function \( \sigma = \sigma(r) \), independent of \( \varepsilon \), such that:

\[
\sigma(r) = \begin{cases} 
1 & \text{for } 0 \leq r \leq 1/2, \\
0 & \text{for } r \geq 3/4,
\end{cases}
\]  

(298)

To study the asymptotic behavior of \( u^\varepsilon \), solution of (296), we propose an asymptotic expansion of the following type:

\[
u^\varepsilon \approx \sum_{j=0}^{\infty} \varepsilon^j (u^j + \Theta^j).
\]  

(299)

Here, at each order of \( \varepsilon^j, j \geq 0 \), \( u^j \) corresponds to the outer expansion (outside of the boundary layer) of \( u^\varepsilon \) as explained below. To balance the discrepancy on the boundary \( \partial \Omega \) of \( u^\varepsilon \) and of the \( u^j, 0 \leq j \leq n \), we introduce the correctors \( \Theta^j \), \( 0 \leq j \leq n \), which will contribute mainly inside of the boundary layer: \( \Theta^j \) will be itself the sum of several boundary layer functions as we shall see.

To determine the asymptotic expansion (299), we start with the outer expansion for \( u^\varepsilon \), \( u^\varepsilon \approx \sum_{j=0}^{\infty} \varepsilon^j u^j \). Inserting formally this expansion into (296), we find the equations for the \( u^j \):

\[
-\frac{\partial u^0}{\partial x} = f, \quad -\frac{\partial u^j}{\partial x} = \Delta u^{j-1}, \quad j \geq 1.
\]  

(300)

We supplement these equations with the inflow zero boundary conditions. Integrating the equations in \( x \), we find the smooth outer solutions \( u^j \) in the form,

\[
u^0 = \int_0^1 f(x_1, y) \, dx_1, \quad u^j = \int_0^1 \Delta u^{j-1}(x_1, y) \, dx_1, \quad j \geq 1.
\]  

(301)

For simplicity in the analysis below, we first decompose

\[
f = f_1 + f_2, \text{ with } f_1 = \sigma(y) f, \ f_2 = (1 - \sigma(y)) f.
\]  

(302)
Thanks to the symmetry and by superposition, it suffices to consider Eq. (296) with $f = f_1$. Thus, we may assume that $f$ is infinitely flat at $y = 1$, i.e. $\partial_y f = 0$ at $y = 1$ for all $\alpha \geq 0$. Hence, we notice that the $u^j$'s satisfy the boundary condition (296)$_2$ along the edges $x = 1$ and $y = 1$, and not on the two other sides of $\partial \Omega$, $y = 0$, and $x = 0$. Hence we expect boundary layers to occur near those edges, and they are constructed as

$$\Theta^j = \varphi^j + \xi^j + \theta^j + \zeta^j, \quad j \geq 0,$$

where

$\varphi^j$ is the parabolic boundary layer near $y = 0$,

$\xi^j$ is the elliptic boundary layer resolving the compatibility issues, in the construction of $\varphi^j$, at the corners $(1,0)$,

$\theta^j$ is the ordinary boundary layer near $x = 0$,

$\zeta^j$ is the ordinary corner layer which manages the effect of $\varphi^j$ along $x = 0$.

### 4.1. Boundary layer analysis at order $\varepsilon^0$.

Following the construction of the boundary layers suggested in [42], we list them briefly below.

**Parabolic boundary layers (PBL)**

At the bottom boundary, i.e., at $y = 0$, in general $u^\varepsilon - u^0 = -u^0 \neq 0$. To resolve this difficulty, we construct below the parabolic boundary layer corrector $\bar{\varphi}^0$. Using the stretched variable $\bar{y} = y/\sqrt{\varepsilon}$, we find the dominating differential operators which lead to the parabolic boundary layer corrector. For $\bar{\varphi}^0 = \bar{\varphi}^0(x, \bar{y})$, $0 \leq j \leq n$,

$$\begin{cases}
-\varepsilon \frac{\partial^2 \bar{\varphi}^0}{\partial \bar{y}^2} - \frac{\partial \bar{\varphi}^0}{\partial x} = 0 & \text{for } (\bar{y}, x) \in \mathbb{R}^+ \times (0, 1), \\
\bar{\varphi}^0 = -u^0(x, 0) & \text{at } y = 0, \\
\bar{\varphi}^0 \to 0 & \text{as } \bar{y} \to \infty, \\
\bar{\varphi}^0 = 0 & \text{at } x = 1.
\end{cases}$$

From [11], [71], [39] or [122], we recall the explicit expressions of $\bar{\varphi}^0 = \varphi^0(x, \bar{y})$,

$$\varphi^0 = -\sqrt{\frac{2}{\pi}} \int_{\bar{y}/\sqrt{2(1-x)}}^{\infty} \exp \left( -\frac{y_1^2}{2} \right) u^0(x + \frac{\bar{y}^2}{2y_1}, 0) dy_1,$$

Since $u^0(1,0) = 0$, some pointwise and $L^p$ estimates for the $\varphi^0$ can be deduced (see the Appendix in [42]).

**Lemma 4.1.** For each $i, m, 0 \leq i + m \leq 1$ and $s \geq 0$, we have the estimates:

$$\left| y^s \frac{\partial^{i+m} \varphi^0}{\partial x^i \partial y^m} \right| \leq \kappa \varepsilon^{-\frac{s}{2}} \exp \left( -c \frac{y}{\sqrt{\varepsilon}} \right), \text{ p.t.w.,}$$

and

$$\left\| y^s \frac{\partial^{i+m} \varphi^0}{\partial x^i \partial y^m} \right\|_{L^p(\Omega)} \leq \kappa \varepsilon^{-\frac{s}{2} + \frac{1}{p}},$$

for a generic constant $c > 0$ independent of $x$, $y$ and $\varepsilon$. 

Ordinary boundary layers (OBL)  
We note that in general
\[ u^\varepsilon - u^0 - \varphi^0 = -u^0 - \varphi^0 \neq 0 \text{ at } x = 0. \]
To resolve this difficulty, we first handle the value $-u^0$ at $x = 0$ by the so-called ordinary boundary layer $\bar{\vartheta}^0$ introduced in this section. Using the stretched variable $\bar{x} = x/\varepsilon$, we find the equations of $\bar{\vartheta}^0 = \bar{\vartheta}^0(\bar{x}, y)$:

\[
\begin{cases}
-\varepsilon \frac{\partial^2 \bar{\vartheta}^0}{\partial x^2} - \frac{\partial \bar{\vartheta}^0}{\partial x} = 0 \text{ in } \Omega, \\
\bar{\vartheta}^0 = -u^0(0, y) \text{ at } x = 0, \\
\bar{\vartheta}^0 \to 0 \text{ as } \bar{x} \to \infty.
\end{cases}
\]  

The explicit expression of $\bar{\vartheta}^0$ is readily available,
\[ \bar{\vartheta}^0 = -u^0(0, y) \exp\left(-\frac{x}{\varepsilon}\right), \]  
and we easily deduce the following estimates.

**Lemma 4.2.** We have the estimates:

\[
\left| x^r \frac{\partial^{i+m} \bar{\vartheta}^0}{\partial x^i \partial y^m} \right| \leq \kappa \varepsilon^{-i} \exp\left(-c\frac{x}{\varepsilon}\right), \quad r, i, m \geq 0, \text{ p.t.w.,}
\]  
and

\[
\left\| x^r \frac{\partial^{i+m} \bar{\vartheta}^0}{\partial x^i \partial y^m} \right\|_{L^p(\Omega)} \leq \kappa \varepsilon^{-i+\frac{2}{p}+\frac{1}{p}}, \quad r, i, m \geq 0.
\]

for a constant $c$ independent of $x$, $y$ and $\varepsilon$.

Ordinary corner layers (OCL)  
We next handle the value $-\bar{\varphi}^0$ at $x = 0$ by introducing the ordinary corner layer correctors $\bar{\zeta}^0$. Using the stretched variables $\bar{x} = x/\varepsilon$ and $\bar{y} = y/\sqrt{\varepsilon}$ near $(0, 0)$, we find the equations for $\bar{\zeta}^0 = \bar{\zeta}^0(\bar{x}, \bar{y})$:

\[
\begin{cases}
-\varepsilon \frac{\partial^2 \bar{\zeta}^0}{\partial x^2} - \frac{\partial \bar{\zeta}^0}{\partial x} = 0 \text{ in } \Omega, \\
\bar{\zeta}^0 = -\varphi^0(0, \frac{y}{\sqrt{\varepsilon}}) \text{ at } x = 0, \\
\bar{\zeta}^0 \to 0 \text{ as } \bar{x} \to \infty.
\end{cases}
\]  

The explicit solution $\bar{\zeta}^0$ is readily available,
\[ \bar{\zeta}^0 = -\varphi^0(0, \frac{y}{\sqrt{\varepsilon}}) \exp\left(-\frac{x}{\varepsilon}\right), \]  
and we easily deduce the following estimates.

**Lemma 4.3.** For $0 \leq i + m \leq 1$, and $r, s \geq 0$, we have the estimates:

\[
\left| x^r y^s \frac{\partial^{i+m} \bar{\zeta}^0}{\partial x^i \partial y^m} \right| \leq \kappa \varepsilon^{-i+s-m} \exp\left(-c\left(\frac{x}{\varepsilon} + \frac{y}{\sqrt{\varepsilon}}\right)\right), \text{ p.t.w.,}
\]  
and

\[
\left\| x^r y^s \frac{\partial^{i+m} \bar{\zeta}^0}{\partial x^i \partial y^m} \right\|_{L^p(\Omega)} \leq \kappa \varepsilon^{-i+s-m+\frac{2}{p}},
\]

for a generic constant $c > 0$ independent of $x$, $y$ and $\varepsilon$. 
Now, we finally obtain that
\[
u^\varepsilon - (u^0 + \varphi^0 + \theta^0 + \zeta^0) = \begin{cases}
0 & \text{at } x = 0, \\
\bar{\theta}^0 + \bar{\zeta}^0 = \text{e.s.t.} & \text{at } x = 1, \\
\bar{\theta}^0 + \bar{\zeta}^0 = (u^0 - \varphi^0)(0, 0) \exp \left( - \frac{x}{\varepsilon} \right) = 0 & \text{at } y = 0, \\
\bar{\varphi}^0 = \text{e.s.t.} & \text{at } y = 1.
\end{cases}
\] (316)

Hence, with the boundary layers \( \varphi^0, \theta^0, \zeta^0 \), we resolve all the discrepancies between \( u^0 \) and the zero boundary condition except for the e.s.t. terms.

To handle the e.s.t. on \( \partial \Omega \), here and after as indicated in (316), using the smooth cut-off functions \( \sigma(x), \sigma(y) \) as in (298), we define
\[
\Theta^0 = \varphi^0 + \theta^0 + \zeta^0, \quad \Theta^0 = \varphi^0 + \theta^0 + \zeta^0,
\]
where \( \varphi^0 = \sigma(y)\varphi^0, \theta^0 = \sigma(x)\theta^0 \), and \( \zeta^0 = \sigma(x)\zeta^0 \) as in (312) but with \( \varphi^0(0, y/\varepsilon) \) being replaced by \( \varphi^0(0, y/\sqrt{\varepsilon}) \). We then set
\[
0 = w_{\varepsilon, 0} = u^\varepsilon - (u^0 + \Theta^0), \quad \bar{w}_{\varepsilon, 0} = u^\varepsilon - (u^0 + \Theta^0).
\] (318)

Let \( L_\varepsilon u = -\varepsilon \Delta u - \partial u / \partial x \). Noting that \( w_{\varepsilon, 0} = 0 \) on \( \partial \Omega \), the equation of \( w_{\varepsilon, 0} \) now reads:
\[
\begin{cases}
L_\varepsilon w_{\varepsilon, 0} = \varepsilon \left( \Delta u^0 + \frac{\partial^2 \varphi^0}{\partial x^2} + \frac{\partial^2 \theta^0}{\partial y^2} + \frac{\partial^2 \zeta^0}{\partial y^2} \right) + L_\varepsilon (w_{\varepsilon, 0} - \bar{w}_{\varepsilon, 0}) & \text{in } \Omega, \\
0 & \text{on } \partial \Omega.
\end{cases}
\] (319)

Observing that
\[
\|w_{\varepsilon, 0} - \bar{w}_{\varepsilon, 0}\|_{H^1} \leq \kappa \exp \left( - \frac{c}{\sqrt{\varepsilon}} \right), \quad c > 0,
\] (320)
we thus have
\[
\left| \int_\Omega L_\varepsilon (w_{\varepsilon, 0} - \bar{w}_{\varepsilon, 0}) w_{\varepsilon, 0} dxdy \right| \leq \kappa \exp \left( - \frac{c}{\sqrt{\varepsilon}} \right) \|w_{\varepsilon, 0}\|_{H^1}, \quad c > 0.
\] (321)

Hence, this quantity will be absorbed by other norms and we will omit it.

We now multiply (354) by \( e^\varepsilon w_{\varepsilon, 0} \) and integrate over \( \Omega \). Integrating by parts and using Lemmas 4.1, 4.2 and 4.3, we find that
\[
\varepsilon \|w_{\varepsilon, 0}\|_{H^1}^2 + \frac{1 - \varepsilon}{2} \|w_{\varepsilon, 0}\|_{L^2}^2 \\
\leq \varepsilon \int_\Omega |\Delta u^0||w_{\varepsilon, 0}| dxdy \\
+ \varepsilon \int_\Omega \left| \frac{\partial \varphi^0}{\partial x} + \frac{\partial \theta^0}{\partial y} + \frac{\partial \zeta^0}{\partial y} \right| |\nabla w_{\varepsilon, 0}| dxdy \\
\leq \kappa \varepsilon + \frac{1}{4} \|w_{\varepsilon, 0}\|_{L^2}^2 + \varepsilon \|w_{\varepsilon, 0}\|_{H^1}^2,
\] (322)

If we impose one compatibility condition at the inflow, i.e. \( f(1, 0) = 0 \), the error estimate above is improved. If \( f(1, 0) = 0 \), we note that \( u^0(1, 0) = (\partial u^0 / \partial x)(1, 0) = 0 \). Then, Lemmas 4.1 and 4.3 hold for \( 0 \leq i + m \leq 2 \) (see Lemmas 4.4 and 4.6 below.
and see also the Appendix in [42]). Using the Hardy inequality (see, e.g., [63]) we then find that

\[
\begin{align*}
\varepsilon\|w_{\varepsilon,0}\|_{H^1}^2 + \frac{1 - \varepsilon}{2}\|w_{\varepsilon,0}\|_{L^2}^2 & \\
& \leq \kappa\varepsilon \int_{\Omega} \left( \Delta u^0 + \frac{\partial^2 \varphi^0}{\partial x^2} + \frac{\partial^2 \varphi^0}{\partial y^2} \right) w_{\varepsilon,0} \, dx \, dy \\
& \quad + \kappa\varepsilon \int_{\Omega} \left( \frac{\partial^2 \zeta^0}{\partial y^2} \right) \left( \frac{w_{\varepsilon,0}}{x} \right) \, dx \, dy \\
& \leq \kappa\varepsilon^2 + \frac{1}{4}\|w_{\varepsilon,0}\|_{L^2}^2 + \frac{\varepsilon}{2}\|w_{\varepsilon,0}\|_{H^1}^2.
\end{align*}
\]

(323)

The \(H^2\) estimates follow from the elliptic regularity theory.

The above errors were derived when \(f\) was infinitely flat at \(y = 1\), and hence we have constructed boundary and corner layers \(\varphi^0 = \varphi^0_B, \zeta^0 = \zeta^0_B\) only at the bottom edge along \(y = 0\). For a general \(f\), along the top edge, \(y = 1\), the correctors are similarly constructed and denoted respectively by \(\varphi^0_T\) and \(\zeta^0_T\) (see Figure 1).

Thanks to the superposition of solutions (see (302) above), we conclude the following error estimates.

**Theorem 4.1.** Let \(u^\varepsilon\) be the solution of Eq. (296). Without any compatibility conditions on \(f\), we have

\[
\left\| u^\varepsilon - (u^0 + \varphi^0_B + \varphi^0_T + \theta^0 + \zeta^0_B + \zeta^0_T) \right\|_\varepsilon \leq \kappa\varepsilon^3.
\]

(324)

With the compatibility conditions,

\[
f(1,0) = f(1,1) = 0,
\]

we have

\[
\left\| u^\varepsilon - (u^0 + \varphi^0_B + \varphi^0_T + \theta^0 + \zeta^0_B + \zeta^0_T) \right\|_\varepsilon \leq \kappa\varepsilon,
\]

(325)

\[
\left\| u^\varepsilon - (u^0 + \varphi^0_B + \varphi^0_T + \theta^0 + \zeta^0_B + \zeta^0_T) \right\|_{H^2(\Omega)} \leq \kappa\varepsilon^{-\frac{1}{2}}.
\]

(326)

**Remark 4.1.** Without any compatibility condition, we will need the elliptic boundary layers \(\xi^0 = \zeta^0_B + \zeta^0_T\) as in Theorem 4.2 for \(n = 0\) below.

4.2. **Boundary layer analysis at arbitrary order** \(\varepsilon^n, n \geq 0\). For simplicity in the analysis, we assume that \(f\) is infinitely flat at \(y = 1\). We now construct high order boundary layers \(\Theta^j\) so that \(u^\varepsilon - \sum_{j=0}^{n} \varepsilon^j (u^j + \Theta^j)\) is small.

**Parabolic boundary layers (PBL)**

At the bottom boundary, i.e., at \(y = 0\), in general \(-u^j \neq 0\). To resolve this difficulty, we first construct the parabolic boundary layer correctors \(\tilde{\varphi}^j\). We formally insert the asymptotic expansion \(u^\varepsilon \cong \sum_{j=0}^{\infty} \varepsilon^j \tilde{\varphi}^j\) into (296). Using the stretched variable \(\bar{y} = y/\sqrt{\varepsilon}\) and collecting the terms of the same order \(\varepsilon^j\), we find the equations of the parabolic boundary layer correctors:

\[
-\varepsilon \frac{\partial^2 \tilde{\varphi}^j}{\partial y^2} - \frac{\partial \tilde{\varphi}^j}{\partial x} = \frac{\partial^2 \tilde{\varphi}^{j-1}}{\partial x^2} \quad \text{in} \ \Omega,
\]

(327)

where we set \(\tilde{\varphi}^{-1} \equiv 0\).
To resolve the compatibility issues for the parabolic boundary layers at the inflow corner \((1,0)\), using the smooth cut-off function \(\sigma(r)\) as in (298), we set
\[
\gamma^j(x) = \tilde{\gamma}^j(x) \sigma(1-x),
\]
where
\[
\tilde{\gamma}^j(x) = -\sum_{i=1}^{2n+1-2j} \frac{(x-1)^i}{i!} \frac{\partial^i u^j}{\partial x^i}(1,0), \quad 0 \leq j \leq n.
\]

The cut-off suppresses additional discrepancies at the outflow which can lead to the so-called elliptic corner layers (see [122]). Then, introducing the stretched variable \(\bar{y} = y/\sqrt{\varepsilon}\), the equation (327) is written as well as a compatible boundary condition as follows. For \(\bar{\varphi}^j = \bar{\varphi}^j(x,\bar{y}), 0 \leq j \leq n\),
\[
\begin{cases}
-\varepsilon \frac{\partial^2 \bar{\varphi}^j}{\partial y^2} - \frac{\partial \bar{\varphi}^j}{\partial x} = \frac{\partial^2 \bar{\varphi}^{j-1}}{\partial x^2} & \text{for } (\bar{y},x) \in \mathbb{R}^+ \times (0,1),
\bar{\varphi}^j = g^j(x) := -u^j(x,0) - \gamma^j(x) & \text{at } y = 0,
\bar{\varphi}^j \rightarrow 0 & \text{as } \bar{y} \rightarrow \infty,
\bar{\varphi}^j = 0 & \text{at } x = 1.
\end{cases}
\]

Note that the smooth boundary conditions \(g^j(x), 0 \leq j \leq n\), along \(y = 0\) are compatible with the other zero boundary conditions at \(x = 1\) in the sense that, for each \(0 \leq j \leq n\),
\[
\frac{\partial^i g^j}{\partial x^i}(1) = 0, \quad 0 \leq i \leq 2n + 1 - 2j.
\]

From [11], [71], [39] or [122], we recall the explicit expressions of \(\bar{\varphi}^j = \bar{\varphi}^j(x,\bar{y}), 0 \leq j \leq n\),
\[
\varphi^0 = \sqrt{\frac{2}{\pi}} \int_{\bar{y}/\sqrt{2(1-x_1)}}^{\infty} \exp \left( -\frac{y_1^2}{2} \right) g^0(x + \frac{y_1^2}{2y_1^2}) dy_1,
\]

**Figure 1.** Location of the outer solution \(u^j\) and the boundary layer correctors.
To account for the discrepancy of $\bar{\varphi}^j$ and $\gamma^j$, we introduced the boundary conditions, by parts. To cancel that in \((134)\) or \((135)\) with \((216)\) by using a change of variable and integrating by parts.

Some pointwise and $L^p$ estimates for the $\bar{\varphi}^j$, $0 \leq j \leq n$ can be deduced (see the Appendix in [42]):

**Lemma 4.4.** For each $0 \leq j \leq n$, $0 \leq i + m \leq 2n + 2 - 2j$ and $s \geq 0$, we have the estimates:

$$
|y^s \frac{\partial^{i+m} \bar{\varphi}^j}{\partial x^i \partial y^m}| \leq \kappa \varepsilon^{\frac{s-m}{2}} \exp\left(-c \frac{y}{\sqrt{\varepsilon}}\right), \text{ p.t.w.,} \tag{333}
$$

and

$$
\left\|y^s \frac{\partial^{i+m} \bar{\varphi}^j}{\partial x^i \partial y^m}\right\|_{L^p(\Omega)} \leq \kappa \varepsilon^{\frac{s-m}{2} + \frac{1}{p}}, \tag{334}
$$

for a generic constant $c > 0$ independent of $x$, $y$ and $\varepsilon$.

**Elliptic boundary layers (EBL)**

In Section 4.2, to construct the consistent parabolic boundary layer correctors $\bar{\varphi}^j$, we considered the $\gamma^j$ above. To cancel $\gamma^j$ along $y = 0$, we introduce the elliptic boundary layers $\xi^j$:

$$
\begin{aligned}
-\varepsilon \frac{\partial^2 \xi^j}{\partial x^2} - \varepsilon \frac{\partial \xi^j}{\partial y} - \frac{\partial \xi^j}{\partial x} &= 0 \quad \text{in } \Omega, \\
\xi^j &= \gamma^j(x) \quad \text{at } y = 0, \\
\xi^j &= 0 \quad \text{at } x = 0, 1, \text{ or } y = 1.
\end{aligned} \tag{335}
$$

Performing the energy estimate on \((335)\), we find that

$$
\|\xi^j\|_\varepsilon \leq \kappa \varepsilon^{\frac{1}{4}}, \quad 0 \leq j \leq n. \tag{336}
$$

**Ordinary boundary and corner layers (OBL, OCL)**

Finally, since in general $-u^j \neq 0$ at $x = 0$, we resolve by introducing the so-called ordinary boundary layer $\theta^j$. We insert a formal expansion $u^\varepsilon \approx \sum_{j=0}^\infty \varepsilon^j \bar{\theta}^j$ into the diffusive equation \((296)\). Then, using the stretched variable $\bar{x} = x/\varepsilon$, we find the equations of $\theta^j = \theta^j(\bar{x}, \bar{y})$:

$$
-\varepsilon \frac{\partial^2 \bar{\theta}^j}{\partial x^2} - \frac{\partial \bar{\theta}^j}{\partial y} = \varepsilon^{-1} \frac{\partial^2 \bar{\theta}^{j-2}}{\partial y^2} \quad \text{in } \Omega, \tag{337}
$$

where we set $\bar{\theta}^{-1} = \bar{\theta}^{-2} = 0$. Hence we supplement the equations \((337)\) with the boundary conditions,

$$
\bar{\theta}^j = -u^j(0, y) \quad \text{at } x = 0, \quad \bar{\theta}^j \to 0 \quad \text{as } \bar{x} \to \infty. \tag{338}
$$

To account for the discrepancy of $-\varphi^j$ at $x = 0$, we need to introduce the ordinary corner layers $\tilde{\xi}^j$. We insert a formal expansion $u^\varepsilon \approx \sum_{j=0}^\infty \varepsilon^j \tilde{\xi}^j$ into \((296)\). Then,
using the stretched variables $\bar{x} = x/\varepsilon$ and $\bar{y} = y/\sqrt{\varepsilon}$ near $(0, 0)$, we collect the terms of order $\varepsilon^j$. As a result, we find the equations for $\bar{\zeta}^j = \bar{\zeta}^j(\bar{x}, \bar{y})$:

$$
- \varepsilon \frac{\partial^2 \bar{\zeta}^j}{\partial x^2} - \frac{\partial \bar{\zeta}^j}{\partial x} = \bar{\zeta}^{j-1} \text{ in } \Omega.
$$

(339)

Here, we set $\bar{\zeta}^{-1} = 0$. At each order of $\varepsilon^j$, to cancel the error $-\bar{\varphi}^j$ near the corner $(0, 0)$, we supplement the equations (339) with the boundary conditions,

$$
\bar{\zeta}^j = -\bar{\varphi}^j \left(0, \frac{y}{\sqrt{\varepsilon}}\right) \text{ at } x = 0, \quad \bar{\zeta}^j \to 0 \text{ as } \bar{x} \to \infty.
$$

(340)

However, in general $-\bar{\vartheta}^j - \bar{\zeta}^j \neq 0$ at $y = 0$ and hence we expect further discrepancies which lead to the so-called elliptic corner layers near $(0, 0)$ (see [122]). To avoid the use of these elliptic corner layers, we consider at once the discrepancies between $-u^j(0, y) - \bar{\varphi}^j \left(0, y/\sqrt{\varepsilon}\right)$ at $x = 0$ and the zero boundary condition. Combining the equations (337) - (340) and using the formal expansion $u^\varepsilon \cong \sum_{j=0}^{\infty} \varepsilon^j \bar{\vartheta}^j$ with the stretched variable $\bar{x}$, we define the $\vartheta^j$:

$$
\begin{align*}
-\varepsilon \frac{\partial^2 \vartheta^j}{\partial x^2} - \frac{\partial \vartheta^j}{\partial x} &= \frac{\partial^2 \vartheta^{j-1}}{\partial y^2} \text{ in } \Omega, \\
\vartheta^j &= -u^j(0, y) - \bar{\varphi}^j \left(0, \frac{y}{\sqrt{\varepsilon}}\right) \text{ at } x = 0, \\
\vartheta^j &\to 0 \text{ as } \bar{x} \to \infty,
\end{align*}
$$

(341)

where we set $\vartheta^{-1} = 0$. The explicit solutions $\vartheta^j$ of (341) are inductively found to be of the form,

$$
\vartheta^j = P^j \left(\frac{x}{\varepsilon}, y, \frac{y}{\sqrt{\varepsilon}}\right) \exp \left(-\frac{x}{\varepsilon}\right),
$$

(342)

where $P^j = P^j \left(\frac{x}{\varepsilon}, y, \frac{y}{\sqrt{\varepsilon}}\right) = \sum_{k=0}^j A_k(y) \bar{x}^k$ is a polynomial in $x/\varepsilon$ of degree $j$ whose coefficients are of the form

$$
A_k(y) = \sum_{s=0}^j \lambda_{s,k} \varepsilon^s \frac{\partial^{2s} a_{j-s}}{\partial y^{2s}}(y), \quad k = 0, 1, \ldots, j,
$$

(343)

where $a_k(y) = -u^k(0, y) - \bar{\varphi}^k \left(0, y/\sqrt{\varepsilon}\right)$, $\lambda_{s,k}$ independent of $\varepsilon$, and $P^j(0, y, y/\sqrt{\varepsilon}) = a_j(y) = -u^j(0, y) - \bar{\varphi}^j \left(0, y/\sqrt{\varepsilon}\right)$. Indeed, this expression is obvious for $j = 0$. At order $j$, we assume that the coefficients are of the same form as in (343). At order $j + 1$, from (341)1 using the stretched variable $\bar{x}$ we find that

$$
\varepsilon \frac{\partial^2 \vartheta^{j+1}}{\partial x^2} - \frac{\partial \vartheta^{j+1}}{\partial x} + \varepsilon^2 \frac{\partial^2 \vartheta^j}{\partial y^2} = \sum_{k=0}^j \sum_{s=0}^j \lambda_{s,k} \varepsilon^{s+1} \frac{\partial^{2s+2} a_{j-s}}{\partial y^{2s+2}} (y) \bar{x}^k \exp(-\bar{x}).
$$

(344)

Imposing the boundary condition $\vartheta^{j+1} = a_{j+1}(y) = -u^{j+1}(0, y) - \bar{\varphi}^{j+1} \left(0, y/\sqrt{\varepsilon}\right)$ at $x = 0$ and solving for $\vartheta^j$ in the equation (344), we find that the coefficients are of the form (343) with $j$ replaced by $j + 1$.

We now write $\vartheta^j = \bar{\vartheta}^j + \bar{\zeta}^j$ where

$$
\bar{\vartheta}^j = P^j \left(\frac{x}{\varepsilon}, y\right) \exp \left(-\frac{x}{\varepsilon}\right),
$$

(345)

$$
\bar{\zeta}^j = P_2^j \left(\frac{x}{\varepsilon}, \frac{y}{\sqrt{\varepsilon}}\right) \exp \left(-\frac{x}{\varepsilon}\right),
$$

(346)
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where $P_1^j(x/\varepsilon, y)$ is a polynomial in $x/\varepsilon$ of degree $j$ whose coefficients are

$$\sum_{s=0}^{j} \lambda_s \varepsilon^s (\partial^{2s} u^{1-s}/\partial y^{2s})(0, y),$$

$\lambda_s$ is independent of $\varepsilon$, and $P_1^j(0, y) = -u^j(0, y)$, and $P_2^j(x/\varepsilon, y/\sqrt{\varepsilon})$ is a polynomial in $x/\varepsilon$ of degree $j$ whose coefficients are

$$\sum_{s=0}^{j} \lambda_s \varepsilon^s (\partial^{2s} \bar{v}^{1-s}/\partial y^{2s})(0, y/\sqrt{\varepsilon}),$$

$\lambda_s$ is independent of $\varepsilon$, and $P_2^j(0, y) = -\bar{v}^j(0, y/\sqrt{\varepsilon})$.

Based on (345), one can prove the following estimates (see [42]).

**Lemma 4.5.** For each $0 \leq j \leq n$, we have the estimates:

$$\left| x^r \frac{\partial^{i+m} \bar{v}^j}{\partial x^i \partial y^m} \right| \leq \kappa \varepsilon^{r-i} \exp \left( -c \frac{x}{\varepsilon} \right), \quad r, i, m \geq 0, \text{ p.t.w.},$$

and

$$\left| x^r \frac{\partial^{i+m} \bar{v}^j}{\partial x^i \partial y^m} \right|_{L^p(\Omega)} \leq \kappa \varepsilon^{r-i+m}, \quad r, i, m \geq 0.$$  

for a constant $c$ independent of $x$, $y$ and $\varepsilon$.

Using Lemma 4.4 and (346), we obtain in [42] the estimates below.

**Lemma 4.6.** For each $0 \leq j \leq n$, $0 \leq i + m \leq 2n + 2 - 2j$, and $r, s \geq 0$, we have the estimates:

$$\left| x^r y^s \frac{\partial^{i+m} \bar{v}^j}{\partial x^i \partial y^m} \right| \leq \kappa \varepsilon^{r-i} \frac{x^m}{\varepsilon^m} \exp \left( -c \frac{x}{\varepsilon} + \frac{y}{\sqrt{\varepsilon}} \right), \text{ p.t.w.},$$

and

$$\left| x^r y^s \frac{\partial^{i+m} \bar{v}^j}{\partial x^i \partial y^m} \right|_{L^p(\Omega)} \leq \kappa \varepsilon^{r-i+m} + \frac{1}{p},$$

for a generic constant $c > 0$ independent of $x$, $y$ and $\varepsilon$.

Now, we finally obtain that

$$u^\varepsilon - \sum_{j=0}^{n} \varepsilon^j (u^j + \bar{v}^j + \xi^j + \bar{\theta}^j + \bar{\zeta}^j)$$

\[
= \begin{cases} 
0 & \text{at } x = 0, \\
\sum_{j=0}^{n} \varepsilon^j (\bar{\theta}^j + \bar{\zeta}^j) = \text{e.s.t.} & \text{at } x = 1, \\
\sum_{j=0}^{n} \varepsilon^j (-u^j - \bar{v}^j)(0, 0) \exp \left( -\frac{x}{\varepsilon} \right) = 0 & \text{at } y = 0, \\
\sum_{j=0}^{n} \varepsilon^j \bar{v}^j = \text{e.s.t.} & \text{at } y = 1.
\end{cases}
\]

Hence, with the boundary layers, we resolve all the discrepancies between the $u^j$ and the zero boundary value except for the e.s.t. terms.
As before, using the smooth cut-off functions \( \sigma(x), \sigma(y) \), we define
\[
\Theta^j = \phi^j + \xi^j + \theta^j + \zeta^j, \quad \Theta_i = \phi_i + \xi_i + \theta_i + \zeta_i
\]
where \( \phi^j = \sigma(y) \tilde{\phi}^j, \theta^j = \sigma(x) \tilde{\phi}^j \), as in (345), and \( \zeta^j = \sigma(x) \tilde{\zeta}^j, \zeta_i^j \) as in (346) but with \( \tilde{\phi}^j(0, y/\sqrt{\varepsilon}) \) being replaced by \( \phi^j(0, y/\sqrt{\varepsilon}) \). Then we set
\[
w_{\varepsilon n} = u^\varepsilon - \sum_{j=0}^n \varepsilon^j (u^j + \Theta^j), \quad w_{\varepsilon n} = u^\varepsilon - \sum_{j=0}^n \varepsilon^j (u^j + \Theta_i^j).
\]
We note that \( w_{\varepsilon n} = 0 \) on \( \partial \Omega \). As explained in (320)-(321), we similarly drop the term \( L_\varepsilon(w_{\varepsilon n} - w_{\varepsilon n}) \). From (296), (300), (330) and (341), the equation of \( w_{\varepsilon n} \) then reads:
\[
\begin{aligned}
L_\varepsilon w_{\varepsilon n} &= \varepsilon^{n+1} \left( \Delta u^n + \frac{\partial^2 \tilde{\phi}^n}{\partial x^2} + \frac{\partial^2 \tilde{\theta}^n}{\partial y^2} + \frac{\partial^2 \tilde{\zeta}^n}{\partial y^2} \right) \quad \text{in } \Omega, \\
(\varepsilon n)_{\text{top edge}} &= \phi^j, \xi^j, \text{and } \zeta^j \text{ along the top edge, } y = 1, \text{ denoted respectively by } \tilde{\phi}^j_B, \tilde{\xi}^j_B \text{ and } \tilde{\zeta}^j_B. \quad \text{For a general } \tilde{f}, \text{ as explained before, we similarly construct boundary and corner layers along the top edge, } y = 1, \text{ denoted respectively by } \tilde{\phi}^j_T, \tilde{\xi}^j_T \text{ and } \tilde{\zeta}^j_T \text{ (see Figure 1).}
\end{aligned}
\]
By linearity and superposition of solutions, we conclude the following theorem.

**Theorem 4.2.** Let \( u^\varepsilon \) be the solutions of Eq. (296). Without any compatibility conditions on \( f \), we have for \( n \geq 0 
\]
\[
\| u^\varepsilon - \sum_{j=0}^n \varepsilon^j (u^j + \phi^j + \xi^j + \theta^j + \zeta^j) \|_{H^2(\Omega)} \leq \kappa \varepsilon^{n+1},
\]
and
\[
\| u^\varepsilon - \sum_{j=0}^n \varepsilon^j (u^j + \phi^j + \xi^j + \theta^j + \zeta^j) \|_{H^2(\Omega)} \leq \kappa \varepsilon^{n-\frac{4}{3}},
\]
where \( \phi^j = \phi^j_b + \phi^j_T, \xi^j = \xi^j_b + \xi^j_T \), and \( \zeta^j = \zeta^j_B + \zeta^j_T \).

**Remark 4.2.** When \( f \) satisfies certain compatibility conditions as indicated in (329) so that
\[
\frac{\partial^2 u^j}{\partial x^2}(1, 0) = \frac{\partial^2 u^j}{\partial x^2}(1, 1) = 0, \quad 1 \leq i \leq 2n + 1 - 2j, \quad 0 \leq j \leq n,
\]
the elliptic boundary layers \( \xi^j \) are not needed in Theorem 4.2.
5. Concluding remarks. Some recent progresses on the boundary layer theory are reviewed in this article. As we said in the Introduction, aiming to study the asymptotic behavior of the solutions to some singular perturbation problems, we construct the boundary layer (or interior layer) correctors and obtain the full structural information of

(a) the boundary layers of the reaction-diffusion equation in a smooth curved domain
(b) the boundary and initial layers of the heat equation in a smooth curved domain
(c) the interior layers of the convection-diffusion equation with turning points in an interval domain
(d) the interaction of the boundary and corner layers for the convection-diffusion equation in a rectangular domain

Depending on, e.g., the characteristics of the corresponding limit problems and the geometry of the domain, various types of boundary layer correctors are introduced; the interior layer correctors near the turning point, the ordinary, parabolic, and elliptic boundary layer correctors as well as the initial layer correctors. These correctors successfully balance the discrepancies between the diffusive (or viscous) solution and its limit solution at zero diffusivity (or viscosity) mainly inside of the boundary layers.

Ongoing projects along the same lines as those in this article, we have been generalizing the methodologies introduced here for some equations in fluid mechanics, e.g., the stationary or evolutionary Stokes, (linearized) Navier-Stokes, and (linearized) primitive equations. These current projects will confirm that our method of correctors can be made useful in the analysis and numerical computations of many singular perturbation problems.

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