Local martingale solutions to the stochastic one layer shallow water equations

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We consider the single layer shallow water equations on a bounded domain $M \subset \mathbb{R}^2$ forced by a multiplicative white noise, and obtain the existence and uniqueness of a maximal pathwise solution for a short period of time. The proof relies on the Skorohod representation theorem, the Gyöngy–Krylov theorem, stopping time arguments, and isotropic estimates.

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1. Introduction

In this article, we consider a shallow water equation model that, to the best of our knowledge, has not been studied before. In the deterministic problem, which has been studied extensively, one must assume that the initial data is small or, otherwise, the solution is only known to exist for a short period of time. In the stochastic context we consider the shallow water equations forced by a multiplicative white noise representing e.g. random wind perturbations at the surface and we opt to focus on the latter situation that is we will seek a solution up to a small stopping time. We consider the single layer model; the two layer model will be considered in [21]. In the deterministic context see e.g. [12] and [24], who omit the Coriolis term and assume the external force (denoted by $F$ below) to be zero in the momentum equation. In [4] and [33] the model is similar to that of Orenna et al., but it has an additional term with $\frac{1}{h}$ in the momentum equation. The model most closely related to the present article can be found in [27]. It does include a Coriolis term, but it still assumes no external forcing and it contains the $\frac{1}{h}$ term. For convenience, we choose a model which omits the $\frac{1}{h}$ term and adds the term $-\delta \Delta h$ to the continuity equation in order to absorb some of the terms involving the gradient of the height of the water. Because of this extra viscous term, we require boundary conditions on $h$, that are specified below. We also choose to include an external force that is independent of the solution. A realistic formulation of the external force can be found in e.g. [31], but this adds more unnecessary difficulties to the problem. For more about the physical derivation of these equations, see e.g. [30].

One of the major difficulties inherent to the shallow water problem is that we do not have the cancelation property for the nonlinear term, as is the case of the Navier–Stokes equations, see e.g. [10,11,17–19]. We also do not have the assumption that $v$ is divergence free, as in the Navier–Stokes system (see e.g. [2]). In the deterministic case, this implies that, in general, one can only obtain local in time a priori estimates for the solution, and hence local in time existence of solutions. As we will see below, the same holds in the stochastic context. We know of very few results of local in time existence of solutions of stochastic partial differential equations. Local in time solutions of the Navier–Stokes equations have been obtained in [2]. In this article, the Navier–Stokes equations are reformulated in a functional form due to von Wahl [32] which provides a local solution of an abstract parabolic equation. This result is first technically improved to account for the lack of regularity of the time derivative in the stochastic context. In a second step the mapping defining the solutions is “randomized” to account for a white noise forcing. In partly related directions, we would like to mention the lecture notes [15] in which the author studies the role that white noises may have in preventing blow up. See also [5,6] in which the author derives results of blow-up in finite time for solutions of stochastic pdes. See also [7] in which the authors study the two layer quasi-geostrophic equation; these equations have some similarity with the shallow water equations but, unlike the shallow water equations that we consider, well-posedness is granted for all time in the deterministic context and then in the stochastic context.

Unlike in [2], the point of view that we adopt in this article for the shallow water equation is to leave the equations unchanged and to obtain the stochastic solution by deriving the suitable a priori estimates and exploiting them in the context of the Galerkin method using the usual compactness arguments.

In the theory of stochastic evolution equations two notions of solutions are typically considered namely pathwise (or strong) solutions and martingale (or weak) solutions. In the former notion the driving noise is fixed in advance while in the later case these underlying stochastic elements enter as an unknown
the problem. For more details about the two type solutions, we refer the reader to e.g. [9,14,13,23]. In the study of deterministic nonlinear evolutionary partial differential equations, when the $L^p$ bounds on the time approximation solutions are obtained, the Galerkin approximation scheme will provide us estimates on the time derivatives, and the classical compactness results, such as Aubin–Lions or Arzéla–Ascoli can be applied to help us pass to the limit in the nonlinear terms. However, the classical compactness results cannot extend to the stochastic setting due to the lack of differentiation in time of the solutions. We will utilize a different compactness result based on fractional Sobolev spaces that allows us to treat nonlinear stochastic equations in a way similar to the deterministic case; see [13,28]. Proofs of other compactness embedding theorems can be found in [3,8,26,29].

In this work, we will use the same approach as in [13] and [10] to establish the existence of both martingale and pathwise solutions but our results will provide finite time existence only. We first derive a formal a priori estimate for the original stochastic system assuming that the solutions are sufficiently regular and then we obtain uniform bounds for solutions up to a stopping time. However, in contrast to the deterministic setting, the positiveness of random stopping times are not granted. The absence of lower bound on the stopping times leads to further difficulties later on when deriving the compactness result and passing to the limit. To circumvent these difficulties, we introduce the modified system which truncates the nonlinear terms. We then derive the existence of global martingale solutions for this system by using the Prokhorov theorem, which is used to obtain the compactness results for the sequence of probability measures associated with the approximate solutions. We then upgrade convergence in distributions to almost sure convergences relative to the new underlying stochastic basis by employing the Skorohod embedding theorem. To deduce the existence of global pathwise solutions for the system relative to the initial stochastic basis, we will employ the Gyöngy–Krylov theorem which is the infinite dimensional version of the classical Yamada–Watanabe theorem (see e.g. [25]). Consequently, we derive the existence of both local martingale and pathwise solutions for the original stochastic system by introducing an appropriate positive stopping time afterward.

This article is organized as follows: In Section 2, we introduce the function spaces (Section 2.1) as well as the deterministic and stochastic frameworks (Section 2.2). In Section 3, we provide formal a priori estimates on the original system. In Section 4, we introduce a Galerkin scheme for the modified system, and by making use of an appropriate cut-off function, we are able to establish uniform a priori estimates for the corresponding sequence of approximate solutions. These estimates are used to develop the compactness argument, then with an application of the Skorokhod embedding theorem, which leads to strong convergence of some subsequence, we obtain the global existence of martingale solutions to the modified system. In this section, we also prove the pathwise uniqueness of global martingale solutions and by an application of the Gyöngy–Krylov Theorem we deduce the global existence of pathwise solutions. In Section 5, we establish the existence of both local martingale solutions pathwise solutions and maximal pathwise solution by defining an appropriate stopping time. Finally in the Appendices, we present some measurability results and the adapted stochastic Gronwall lemma, among the other existing results used in the article.

We will consider the following system, where $\mathcal{M} \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary $\partial \mathcal{M}$ and $T \in (0, \infty)$:

$$
d\mathbf{v} + (-\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + g\nabla h + f\mathbf{k} \times \mathbf{v})dt = Fdt + \sigma_1(\mathbf{v}, h)dB_1 \quad \text{in } \mathcal{M} \times (0, T),$$  \hspace{1cm} (1.1a)

$$dh + (-\delta \Delta h + \nabla \cdot (h\mathbf{v}))dt = \sigma_2(\mathbf{v}, h)dB_2 \quad \text{in } \mathcal{M} \times (0, T),$$  \hspace{1cm} (1.1b)

with initial conditions:

$$
\mathbf{v}(t = 0) = \mathbf{v}_0(x, y) \quad \text{in } \mathcal{M},$$  \hspace{1cm} (1.1c)

$$h(t = 0) = h_0(x, y) > 0 \quad \text{in } \mathcal{M},$$  \hspace{1cm} (1.1d)
and we assume, for instance, the Dirichlet boundary conditions:

\begin{align}
\mathbf{v} &= 0 \quad \text{on } \partial \mathcal{M} \times (0, T), \\
\dot{h} &= 0 \quad \text{on } \partial \mathcal{M} \times (0, T). 
\end{align}

\tag{1.1e} \tag{1.1f}

Here, \(\mathbf{v} = (u, v)\) where \(u := u(x, y, \omega, t)\) is the velocity of the water in the \(x\) direction and \(v := v(x, y, \omega, t)\) is the velocity of the water in the \(y\) direction; \(h := h(x, y, \omega, t)\) is the depth of the water, but we will assume that \(h = H + \dot{h}\) where \(H > 0\) is the average depth of the water, a constant, and \(\dot{h}\) is how much the height of the water deviates from its average depth. Also, \(\nu\) is the viscosity, \(\delta\) is a viscosity like positive constant, \(g\) is the gravitational constant, \(f\) is the Coriolis parameter assumed to be constant, \(F := F(x, y, t), \mathbf{v}_0(x, y)\) and \(h_0(x, y)\) are given. Here, we assume \(F\) to be deterministic so as to avoid technicalities when we prove the existence of martingale solutions. For this model, we will impose certain usual Lipschitz conditions on \(\sigma_1\) and \(\sigma_2\), both of which are given, and \(W_i, i = 1, 2\), are cylindrical Wiener processes specified in Section 2.2 below. Equations (1.1a) and (1.1b) are correctly defined using Itô integrals as explained below.

**Remark 1.1.** Different boundary conditions on \(\mathbf{v}\) and \(h\) appear also in the literature, such as:

\begin{align}
\mathbf{v} \cdot n &= 0 \quad \text{and} \quad \text{curl}(\mathbf{v}) = 0 \quad \text{on } \partial \mathcal{M} \times (0, T), \\
\nabla \dot{h} \cdot n &= 0 \quad \text{on } \partial \mathcal{M} \times (0, T). 
\end{align}

\tag{1.2} \tag{1.3}

This set of boundary conditions yields the same type of results but it requires more technical work.

**Remark 1.2.** Physically, the depth of the water is necessarily positive. For a proof of the positivity of \(h\) in the deterministic case, see Appendix B.

2. Analytic tools

In this section, we collect and define the deterministic and stochastic tools needed throughout this article.

2.1. Function spaces

We will work in the spaces \(H = H_1 \times H_2, V = V_1 \times V_2\) where

\[ H_1 := L^2(\mathcal{M})^2, \quad V_1 := (H_0^1(\mathcal{M}))^2, \quad H_2 := L^2(\mathcal{M}), \quad V_2 := H_0^1(\mathcal{M}). \]

\tag{2.1}

On \(H_1\) and \(H_2\), we will use the typical \(L^2\)-inner product and norm denoted by \((\cdot, \cdot)\) and \(|\cdot|\), respectively, while on \(V_1\) and \(V_2\), we will use \((\cdot, \cdot)\) and \(|\cdot|\), which are the usual \(L^2\)-inner product and norm of the gradients.

We also consider the fractional powers of the \((-\Delta)\) operator with the boundary conditions (1.1e) and (1.1f). Classically, there exists an orthonormal basis \(\{\psi_k\}_{k \geq 1}\) of \(H\) with an unbounded increasing sequence of eigenvalues \(\{\lambda_k\}_{k \geq 1}\) such that \(-\Delta \psi_k = \lambda_k \psi_k\). We have \(D(-\Delta) = V \cap (H^2(\mathcal{M}))^3\) and for \(\alpha \geq 0\) we define:

\[ D((-\Delta)^{\alpha}) = \left\{ u \in H : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < \infty \right\}, \]

\tag{2.2}

endowed with the Hilbertian norm

\[ |u|_{\alpha} := |(-\Delta)^{\alpha} u| = \left( \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 \right)^{1/2}. \]

\tag{2.3}
Here, $u = \sum_{k=1}^{\infty} u_k \psi_k$ with $|u|^2 = \sum_{k=1}^{\infty} |u_k|^2 < \infty$.

For the Galerkin scheme below, we introduce the finite dimensional spaces $H_n = \text{span}\{\psi_1, \ldots, \psi_n\}$ and let $P_n, Q_n = I - P_n$ be the projection operators in $H$ onto $H_n$ and onto its orthogonal complement. By abuse of notation we will use also the operator $P_n$ to denote $P_n\psi = P_n(\psi, 0)$ and $P_nh = P_n(0, h)$. We have the generalized and reverse Poincaré inequalities which hold for any $\alpha_1 < \alpha_2$:

$$|P_nu|_{\alpha_2} \leq \lambda_n^{\alpha_2-\alpha_1}|P_nu|_{\alpha_1} \quad \text{and} \quad |Q_nu|_{\alpha_1} \leq \frac{1}{\lambda_n^{\alpha_2-\alpha_1}}|Q_nu|_{\alpha_2}. \quad (2.4)$$

2.2. Stochastic framework

In order to define the stochastic terms in (1.1a) and (1.1b), that is $\sigma_1(\psi, h)dW_1$ and $\sigma_2(\psi, h)dW_2$, we first recall some basic notations and notions from stochastic analysis in Hilbert spaces, used here and after. For an extended treatment of this topic, we refer to [9]. Throughout this article, we will work with a given stochastic basis

$$S = \{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k_t\}_{k \geq 1})\}, \quad k = 1, 2,$$

that is a filtered probability space and $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space, $\{\mathcal{F}_t\}_{t \geq 0}$ is a complete, right continuous filtration, $\{W^k_t\}_{k \geq 1}$ is a sequence of independent one dimensional Brownian motions adapted to $\{\mathcal{F}_t\}_{t \geq 0}$.

Let $\mathcal{U}$ be an auxiliary separable real Hilbert space endowed with a Hilbert basis $\{e_j\}_{j \geq 1}$. We then consider $W(t, \cdot, \omega)$ the $\mathcal{U}$-valued stochastic processes, formally represented for the moment, as the following series:

$$W(t, \cdot, \omega) = \sum_{\ell=1}^{\infty} W^\ell(t, \omega)e_{\ell}(\cdot). \quad (2.5)$$

This expression makes each $W$ a cylindrical Brownian motion evolving over a separable space $\mathcal{U}$ with orthogonal basis $e_k$.

Given any pair $(\mathcal{U}, X)$ of separable Hilbert spaces, we denote by

$$L_2(\mathcal{U}, X) := \{R \in \mathcal{L}(\mathcal{U}, X) : \sum_{k=1}^{\infty} |R e_k|^2_X < \infty\} \quad (2.6)$$

the set of Hilbert–Schmidt operators from $\mathcal{U}$ to $X$. This space $L_2(\mathcal{U}, X)$ is a Hilbert space endowed with the following inner product and norm

$$\langle R, S \rangle_{L_2(\mathcal{U}, X)} = \sum_{k=1}^{\infty} \langle R e_k, S e_k \rangle_X \quad \text{and} \quad \|R\|_{L_2(\mathcal{U}, X)}^2 = \sum_{k=1}^{\infty} |R e_k|^2_X. \quad (2.7)$$

It is well-known that the definition above of a Hilbert–Schmidt operator is independent of the choice of the orthonormal basis $e_i$. One may also show that if $R^1 \in L_2(\mathcal{U}, X)$ and $R^2 \in \mathcal{L}(\mathcal{U}, X)$ then $R^2R^1$ and $R^1R^2 \in L_2(\mathcal{U}, X)$.

In this context, we define an auxiliary space $\mathcal{U}_0 \subset \mathcal{U}$ by setting

$$\mathcal{U}_0 := \left\{ u = \sum_{k=1}^{\infty} a_k e_k : \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty \right\},$$
which we endow with the norm
\[ |u|^2_{\mathcal{H}_0} := \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} \]
and observe that the inclusion:
\[ \mathcal{H} \to \mathcal{H}_0 \text{ is Hilbert–Schmidt.} \]

**Assumptions on \( \sigma_i \) for \( i = 1, 2 \)** We shall assume throughout this work that
\[ \sigma_i := \sigma_i(v, h, t, \omega) : H_1 \times H_2 \times [0, T] \times \Omega \to L_2(\mathcal{H}, H_i), \quad (2.8) \]
are measurable, essentially bounded in time and \( L^2 \) in \( \Omega \), \( \{\mathcal{F}_t\}_{t \geq 0} \) adapted and satisfying the following conditions for a.e. \( t \in [0, T] \) and a.s.:
\begin{align}
\|\sigma_i(v, h, t, \omega)\|^2_{L^2(\Omega; H_i)} &\leq K_V (1 + \|v\|^2 + \|h\|^2) \quad (2.9a) \\
\|\sigma_i(v, h, t, \omega)\|^2_{L^2(\Omega; H_i)} &\leq K_H (1 + \|v\|^2 + \|h\|^2) \quad (2.9b) \\
\|\sigma_i(v_1, h_1, t, \omega) - \sigma_i(v_2, h_2, t, \omega)\|^2_{L^2(\Omega; H_i)} &\leq K_V (\|v_1 - v_2\|^2 + \|h_1 - h_2\|^2) \quad (2.9c) \\
\|\sigma_i(v_1, h_1, t, \omega) - \sigma_i(v_2, h_2, t, \omega)\|^2_{L^2(\Omega; H_i)} &\leq K_H (\|v_1 - v_2\|^2 + \|h_1 - h_2\|^2). \quad (2.9d)
\end{align}
For simplicity, we shall sometime write \( \sigma_i(v, h, t, \omega) = \sigma_i(v, h) \).

Finally, given an \( X \)-valued predictable process \( G \in L^2(\Omega; L^2_{\text{loc}}(0, \infty); L_2(\mathcal{H}, X)) \) one may define the (Itô) stochastic integral
\[ M_t := \int_0^t \! G\,dW, \quad (2.10) \]
which belongs to \( \mathcal{M}^2_X \), the space of all \( X \)-valued square integrable martingales (see e.g. [25]).

For a.e. \( t \) and a.s., \( G \in L_2(\mathcal{H}, H) \) so that \( G_k = G \cdot e_k \in H \), where \( \{e_k\} \) is the basis of \( \mathcal{H} \). Then (2.10) can be represented as
\[ M_t = \sum_{k} \int_0^t \! G_k dW_k. \]
The martingale \( \{M_t\}_{t \geq 0} \) has many desirable properties. Most notably for the analysis here, the Burkholder–Davis–Gundy inequality holds which in the present context takes the form,
\[ \mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t \! G\,dW \right|^r_X \right) \leq C_1 \mathbb{E} \left( \int_0^T \! \|G\|^2_{L_2(\mathcal{H}, X)} \,dt \right)^{\frac{r}{2}}, \quad (2.11) \]
valid for \( r \geq 1 \). With \( G_k = G \cdot e_k \), (2.11) can be rewritten as
\[ \mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t \! \sum_{k=1}^{\infty} G_k e_k dW_k \right|^r_X \right) \leq C_1 \mathbb{E} \left( \int_0^T \! \sum_{k=1}^{\infty} \|G_k e_k\|^2_X \,dt \right)^{\frac{r}{2}}, \quad (2.12) \]
Here $C_1$ is an absolute constant depending on $r$. We shall also make use of a variation of inequality (2.11), which applies to fractional derivatives of $M_t$. For $p \geq 2$ and $\alpha \in [0, 1/2)$ we have

$$
E \left( \sup_{t \in [0,T]} \left\| \int_0^t G \, dW \right\|_{W^{\alpha,p}(\Omega;X)}^p \right) \leq C \mathbb{E} \left( \int_0^T \left\| G \right\|_{L_2(\Omega;X)}^p \, dt \right),
$$

which holds for all $X$-valued predictable $G \in L^2(\Omega; L^0_{\text{loc}}([0, \infty); L_2(\Omega, X)))$.

For the convenience of the reader, we shall recall the definition of the spaces $W^{\alpha,p}([0, T]; X)$ in Section 6 below.

We can express (2.13) in a similar form as in (2.12) as

$$
E \left( \sup_{t \in [0,T]} \left| \sum_k \int_0^t G e_k \, dW^k \right|_{W^{\alpha,p}(\Omega;X)}^p \right) \leq C \mathbb{E} \left( \int_0^T \sum_k \left| Ge_k \right|^p_{X} \, dt \right).
$$

We will also make use of the decomposition $u = \sum_{j=1}^{\infty} \xi_j \phi_j$ where $\xi_j = \xi_j(t, \omega)$ and the $\phi_j$ are the eigenvalues of $A = -\Delta$ in $D(A) \subset H$ so that $A u$ becomes $\sum_{j=1}^{\infty} \xi_j \lambda_j \phi_j$; and if $b \in H$, $b = \sum_{j=1}^{\infty} b_j \phi_j$ with $b_j = (b, \phi_j)$.

**Remark 2.1 (Notation).** For $i = 1, 2$, $\sigma_i(v, h, t, \omega) = \sigma_i(v, h)$ as in (2.8) and $W_i = \sum_{k=1}^{\infty} e_k W^k_i$, we have:

$$
\sigma_i(v, h) \, dW_i = \sum_{k=1}^{\infty} \sigma_i^k(v, h) \cdot e_k dW_i^k = \sum_{k, \ell=1}^{\infty} \langle \sigma_i(v, h) e_k, \phi_{\ell} \rangle \phi_{\ell} dW^k_i
$$

$$
= \sum_{k, \ell=1}^{\infty} \sigma_{i,k\ell} \phi_{\ell} dW^k_i,
$$

(2.15)

where

$$
\sigma_i(v, h) \cdot e_k = \sum_{\ell} \sigma_{i,k\ell} \phi_{\ell}, \quad \sigma_{i,k\ell} = \langle \sigma_i(v, h) \cdot e_k, \phi_{\ell} \rangle,
$$

which makes sense since $\sigma_i(v, h) \cdot e_k \in H$ and $\{ \phi_{\ell} \}$ is a Hilbert basis of $H$.

We shall assume furthermore that if $\tilde{v} : [0, T] \times \Omega \to H_1$, $\tilde{h} : [0, T] \times \Omega \to H_2$ are predictable, then so is $\sigma_i(\tilde{v}, \tilde{h})$. Given an $H$-valued predictable process $v \in L^2(\Omega;L^2([0, T];H_1))$, $\bar{h} \in L^2(\Omega; L^2(0,T;H_2))$ the series expansion (2.15) can be shown to be well defined as a stochastic integral and

$$
\left\langle \int_0^r \sigma_1(v, h) \, dW_1, \tilde{v} \right\rangle = \left\langle \sum_k \int_0^r \sigma_{1,k}^k(v, h) \, dW_i^k, \tilde{v} \right\rangle = \sum_k \int_0^r \langle \sigma_{1,k}^k(u, h), \tilde{v} \rangle dW_i^k,
$$

$$
\left\langle \int_0^r \sigma_2(v, h) \, dW_2, \tilde{h} \right\rangle = \left\langle \sum_k \int_0^r \sigma_{2,k}^k(v, h) \, dW_i^k, \tilde{h} \right\rangle = \sum_k \int_0^r \langle \sigma_{2,k}^k(v, h), \tilde{h} \rangle dW_i^k,
$$

(2.16)

for all $\tilde{v} \in H_1$, $\tilde{h} \in H_2$ and stopping time $\tau$. In this context, the two equations (1.1a), (1.1b) fully make sense as Itô integrals with values in the spaces $V_1'$, $V_2'$ after (Itô) integration from 0 to $t$, for a.e. $t \in [0, T]$. 

We will frequently apply the following form of the infinite dimensional version of the Itô formula for \( p = 2 \) or \( p \geq 4 \):

\[
d|v|^p + pν|v|^2 |v|^{p-2} dt = p(F, v) |v|^{p-2} - p((v \cdot \nabla)v, v) |v|^{p-2} dt - pg(\nabla h, v) |v|^{p-2} dt
\]
\[
- p\langle fK, v \rangle |v|^{p-2} dt + \frac{P}{2} \sum_{k=1}^{\infty} |\sigma_k^1(v, h)|^2 |v|^{p-2} dt
\]
\[
+ \frac{p(p-2)}{2} \sum_{k=1}^{\infty} \langle \sigma_k^1(v, h), v \rangle^2 |v|^{p-4} dt + p \sum_{k=1}^{\infty} \langle \sigma_k^1(v, h), v \rangle |v|^{p-2} dW_k^1,
\]

(2.17a)

and

\[
d|h|^p + pδ|h|^2 |h|^{p-2} dt = -p\langle \nabla \cdot (hν), h \rangle |h|^{p-2} dt + \frac{P}{2} \sum_{k=1}^{\infty} |\sigma_k^2(v, h)|^2 |h|^{p-2} dt
\]
\[
+ \frac{p(p-2)}{2} \sum_{k=1}^{\infty} \langle \sigma_k^2(v, h), h \rangle^2 |h|^{p-4} dt + p \sum_{k=1}^{\infty} \langle \sigma_k^2(v, h), h \rangle |h|^{p-2} dW_k^2,
\]

(2.17b)

and we have similar formulas for \( \|u\|^p, \|h\|^p \):

\[
d\|v\|^p + pν\|\Delta v\|^2 |v|^{p-2} dt
\]
\[
= p(F, \Delta v)\|v\|^{p-2} dt - p((v \cdot \nabla)v, v)\|v\|^{p-2} dt - pg(\nabla h, \Delta v)\|v\|^{p-2} dt
\]
\[
- p\langle fK \times v, \Delta v \rangle\|v\|^{p-2} dt + \frac{P}{2} \sum_{k=1}^{\infty} |\sigma_k^1(v, h)|^2 |\Delta v|^{p-2} dt
\]
\[
+ \frac{p(p-2)}{2} \sum_{k=1}^{\infty} \langle \sigma_k^1(v, h), \Delta v \rangle^2 |\Delta v|^{p-4} dt + p \sum_{k=1}^{\infty} \langle \sigma_k^1(v, h), \Delta v \rangle \|\Delta v\|^{p-2} dW_k^1,
\]

(2.18)

\[
d\|h\|^p + pδ\|\Delta h\|^2 \|h\|^{p-2} dt
\]
\[
= -p\langle \nabla \cdot (hν), \Delta h \rangle \|h\|^{p-2} dt + \frac{P}{2} \sum_{k=1}^{\infty} |\sigma_k^2(v, h)|^2 \|h\|^{p-2} dt
\]
\[
+ \frac{p(p-2)}{2} \sum_{k=1}^{\infty} \langle \sigma_k^2(v, h), \Delta h \rangle^2 \|\Delta h\|^{p-4} dt + p \sum_{k=1}^{\infty} \langle \sigma_k^2(v, h), \Delta h \rangle \|h\|^{p-2} dW_k^2.
\]

(2.19)

**Remark 2.2.** In (2.17a), (2.17b), (2.18) and (2.19), \( \langle \cdot, \cdot \rangle \) denotes the dual pairing between \( V_1 \) and \( V'_1 \) relative to \( H_1 \), and that between \( V_2 \) and \( (V_2)' \) relative to \( H_2 \).

### 2.3. Definitions of solutions

We now need to define what exactly we mean by a solution to problem (1.1) as in [19]. First, we recall what it means for a stochastic process to be predictable:

**Definition 2.1.** For a given stochastic basis \( S \), let \( \Phi = \Omega \times [0, \infty) \) and take \( G \) to be the \( \sigma \)-algebra generated by the sets of the form

\[
(s, t) \times \mathfrak{F}, \quad 0 \leq s < t < \infty, \quad \mathfrak{F} \in \mathcal{F}_s; \quad \{0\} \times \mathfrak{F}, \quad \mathfrak{F} \in \mathcal{F}_0.
\]

(2.20)

An \( X \)-valued process \( U \) is called **predictable** w.r.t. \( S \) if it is measurable from \( (\Phi, G) \) into \( (X, \mathcal{B}(X)) \) where \( \mathcal{B}(X) \) is the family of Borel sets of \( X \).
We next give the definitions of local and global solutions of (1.1) for both martingale and pathwise solutions. Before that, we make some assumptions for the initial condition \((v_0, h_0)\), which may be random in general. For the case of martingale solutions, since the stochastic basis is unknown, we are only able to specify \((v_0, h_0)\) as an initial probability measure \(\mu_0\) on \(H_1 \times H_2\). For the case of pathwise solutions where the stochastic basis \(S\) is fixed in advance, we assume that, relative to this basis and for \(p \geq 2\), \((v_0, h_0)\) is a \(V_1 \times V_2\) valued random variable such that:

\[
v_0 \in L^p(\Omega, V_1), h_0 \in L^p(\Omega, V_2) \text{ and are } F_0\text{-measurable.} \quad (2.21)
\]

**Definition 2.2 (Local and global martingale solutions).** Suppose that \(\mu_0\) is a probability measure on \(V_1 \times V_2\) such that \(\int |x|^p \, d\mu_0(x) < \infty\) for some \(p\) and suppose that \(F \in L^2(0, T; H_1)\), and for \(i = 1, 2\), \(\sigma_i(v, h)\) satisfies the Lipschitz conditions in (2.9), is predictable, and \(\mathcal{F}_t\)-adapted. Then we say a quadruplet \((\tilde{S}, \tilde{v}, \tilde{h}, \tilde{\tau})\) is a local Martingale solution of problem (1.1) if \(\tilde{S} := \left(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}}, \hat{W}_1, \hat{W}_2\right)\) is a stochastic basis, \(\tilde{\tau}\) is a strictly positive stopping time (i.e. \(\tilde{\tau} > 0\) almost surely) relative to \(\tilde{\mathcal{F}}_t\), and \(\tilde{v}(\cdot \wedge \tilde{\tau}), \tilde{h}(\cdot \wedge \tilde{\tau})\) are \(\tilde{\mathcal{F}}_\tau\)-adapted processes in \(H_1, H_2\), respectively, so that:

\[
\begin{align*}
\tilde{v}(\cdot \wedge \tilde{\tau}) & \in L^2(\tilde{\Omega}; C([0, T]; V_1)), \quad (2.22a) \\
\tilde{h}(\cdot \wedge \tilde{\tau}) & \in L^2(\tilde{\Omega}; C([0, T]; V_2)), \quad (2.22b) \\
\tilde{v}(t) 1_{t \leq \tilde{\tau}} & \in L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))), \quad (2.22c) \\
\tilde{h}(t) 1_{t \leq \tilde{\tau}} & \in L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))). \quad (2.22d)
\end{align*}
\]

Furthermore, the law of \((\tilde{v}(0), \tilde{h}(0))\) is \(\mu_0\), i.e. \(\mu_0(E) = \tilde{\mathbb{P}}((\tilde{v}(0), \tilde{h}(0)) \in E)\) for all Borel subsets \(E \subset H_1 \times H_2\), and \((\tilde{v}, \tilde{h})\) must satisfy almost surely, for every \(t \geq 0\), every \(v \in H_1\), and every \(\eta \in H_2\),

\[
(\tilde{v}(t \wedge \tilde{\tau}), v) + \int_0^{t \wedge \tilde{\tau}} (-\nu \Delta \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{v} + g \nabla \tilde{h} + f k \times \tilde{v} - F, v) \, ds = (\tilde{v}_0, v) + \int_0^{t \wedge \tilde{\tau}} \sum_{k=1}^\infty (\sigma_1(\tilde{v}, \tilde{h})e_k, v) d\tilde{W}^k_1, \quad (2.23)
\]

\[
(\tilde{h}(t \wedge \tilde{\tau}), \eta) + \int_0^{t \wedge \tilde{\tau}} (\nabla \cdot (\tilde{h} \tilde{v}) - \delta \Delta \tilde{h}, \eta) \, ds = (\tilde{h}_0, \eta) + \int_0^{t \wedge \tilde{\tau}} \sum_{k=1}^\infty (\sigma_2(\tilde{v}, \tilde{h})e_k, \eta) d\tilde{W}^k_2. \quad (2.24)
\]

We say that the martingale solution \((\tilde{S}, \tilde{v}, \tilde{h}, \tilde{\tau})\) is global if \(\tilde{\tau} = \infty\) a.s.

**Definition 2.3 (Local, maximal and global pathwise solutions).** Suppose that \(S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_1, W_2)\) is a fixed stochastic basis and that \((v_0, h_0)\) is a \(V_1 \times V_2\) valued random variable (relative to \(S\)) satisfying (2.22) and the same conditions hold for \(F\) and \(\sigma_i, i = 1, 2\).

(i) A triplet \((v, h, \tau)\) is a local pathwise solution to (1.1) if \(\tau\) is a strictly positive stopping time, \(v(\cdot \wedge \tau)\) is an \(\mathcal{F}_\tau\)-adapted process in \(V_1\), and \(h(\cdot \wedge \tau)\) is an \(\mathcal{F}_\tau\)-adapted process in \(V_2\) (relative to the fixed basis \(S\)) such that (2.22)–(2.24) hold.

---

1 One can also assume \(F\) to be random, but we choose \(F\) to be deterministic here, or else it will be unnecessarily tricky for the proof of the existence of the martingale solutions later on.
(ii) Pathwise solutions of (1.1) are said to be unique up to a stopping time \( \tau > 0 \) if given any pair of pathwise solutions \((v_1, h_1, \tau)\) and \((v_2, h_2, \tau)\) which coincide at \( t = 0 \) on a subset \( \Omega_0 \) of \( \Omega \):

\[
\Omega_0 := \{v_1(0) = v_2(0), h_1(0) = h_2(0)\} \subset \Omega,
\]

then

\[
P(\mathbb{1}_{\Omega_0}(v_1(t \land \tau) - v_2(t \land \tau)) = 0, \forall t \geq 0) = 1,
\]

and

\[
P(\mathbb{1}_{\Omega_0}(h_1(t \land \tau) - h_2(t \land \tau)) = 0, \forall t \geq 0) = 1.
\]

(iii) Suppose we have \( \{\tau_n\}_{n \geq 1} \), a strictly increasing sequence of stopping times that converge to a stopping time \( \xi \), and assume that \( v \) and \( h \) are predictable continuous \( \mathcal{F}_t \)-adapted processes in \( H_1 \) and \( H_2 \), respectively. We say that \((v, h, \xi) := (v, h, \xi, \{\tau_n\}_{n \geq 1})\) is a maximal pathwise solution if \((v, h, \tau_n)\) is a local pathwise solution for each \( n \) and

\[
\sup_{t \in [0, \xi]} ||v||^2 + \int_0^\xi |\Delta v|^2 ds + \sup_{t \in [0, \xi]} ||h||^2 + \int_0^\xi |\Delta h|^2 ds = \infty,
\]

a.s. on the set \( \{\xi < \infty\} \). If we have

\[
\sup_{t \in [0, \tau_n]} ||v||^2 + \int_0^{\tau_n} |\Delta v|^2 ds + \sup_{t \in [0, \tau_n]} ||h||^2 + \int_0^{\tau_n} |\Delta h|^2 ds = n,
\]

for almost every \( \omega \in \{\xi < \infty\} \), then the sequence \( \tau_n \) is said to announce a finite blow-up time.

(iv) If \((v, h, \xi)\) is a maximal pathwise solution and \( \xi = \infty \) almost surely, then we say that the solution is global.

We now state the main results in this work:

**Theorem 2.1.** We are given \( \mu_0 \) as a probability measure on \( H_1 \times H_2, F \in L^2(0, T; H_1) \) and \( \sigma_i = \sigma_i(v, h), i = 1, 2 \) satisfying the Lipschitz conditions (2.9), predictable, and \( \mathcal{F}_t \)-adapted. Then there exists a local martingale solution \((S, \tilde{v}, h, \tau)\) to (1.1).

**Theorem 2.2.** Assume we are working relative to a given fixed stochastic basis \( \mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_1, W_2) \). Suppose furthermore that (2.21) also holds. Then there exists a unique, maximal pathwise solution \((v, h, \xi, (\tau_n)_{n \geq 1})\) to (1.1).

The strategy to prove both of these theorems is that we will consider a modified system with a cut-off function damping the non-linear term, so that we have global existence of martingale solutions and pathwise solutions for this modified system using the Galerkin approximation. We then return to the original system (1.1) by introducing a stopping time which will be proven to be positive.

3. Formal a priori estimates

We first prove some a priori estimates on the solutions of this system, assuming all of the functions are smooth.
Adding Itô's formulas (2.18) and (2.19), we obtain
\[
d\|v\|^2 + 2\nu |\Delta v|^2 dt + d\|h\|^2 + 2d |\Delta h|^2 dt \\
= 2\langle F, \Delta v \rangle dt - 2g(\nabla h, \Delta v) dt - 2(\nabla \cdot v, \Delta v) dt - 2(\nabla \cdot (h v), \Delta h) dt \\
+ \sum_{k=1}^{\infty} \|\sigma_1(v, h)e_k\|^2_{V_1} dt + \sum_{k=1}^{\infty} \|\sigma_2(v, h)e_k\|^2_{V_2} dt \\
+ 2 \sum_{k=1}^{\infty} \langle \sigma_1(v, h)e_k, \Delta v \rangle dW_1^k + 2 \sum_{k=1}^{\infty} \langle \sigma_2(v, h)e_k, \Delta h \rangle dW_2^k. 
\]

(3.1)

We integrate (3.1) in time over \([0, r]\), for \(0 \leq r \leq s \leq T\), take the supremum in \(r\) over \([0, s]\); we deduce that
\[
\sup_{r \in [0,s]} \|v(r)\|^2 + 2\nu \int_0^s |\Delta v|^2 dt + \sup_{r \in [0,s]} \|h(r)\|^2 + 2\delta \int_0^s |\Delta h|^2 dt \\
\leq 4\|v_0\|^2 + 4\|h_0\|^2 + 8 \int_0^s |\langle F, \Delta v \rangle| dt + 8g \int_0^s |\langle \nabla h, \Delta v \rangle| dt + 8 \int_0^s |\langle v \cdot \nabla \Delta v \rangle| dt \\
+ 8 \int_0^s |\langle \nabla (h v), \Delta v \rangle| dt + 8 \int_0^s |\langle f k \times v, \Delta v \rangle| dt \\
+ 4 \int_0^s \sum_{k=1}^{\infty} \|\sigma_1(v, h)e_k\|^2_{V_1} dt + 4 \int_0^s \sum_{k=1}^{\infty} \|\sigma_2(v, h)e_k\|^2_{V_2} dt \\
+ 8 \sup_{0 \leq r \leq s} \left| \int_0^r \sum_{k=1}^{\infty} \langle \sigma_1(v, h)e_k, \Delta v \rangle dW_1^k \right| + 8 \sup_{0 \leq r \leq s} \left| \int_0^r \sum_{k=1}^{\infty} \langle \sigma_2(v, h)e_k, \Delta h \rangle dW_2^k \right| \\
= 4\|v_0\|^2 + 4\|h_0\|^2 + \sum_{r=1}^9 J_r. 
\]

(3.2)

By simply using the Cauchy–Schwarz inequality, we obtain the estimate for \(J_1\) and \(J_2\) as follows:
\[
J_1 = 8 \int_0^s |\langle F, \Delta v \rangle| dt \leq \frac{20}{\nu} \int_0^s |F|^2 dt + \frac{\nu}{5} \int_0^s |\Delta v|^2 dt. 
\]

(3.3)
\[
J_2 = 8 \int_0^s |\langle \nabla h, \Delta v \rangle| dt \leq \frac{20}{\nu} \int_0^s \|h(r)\|^2 + \frac{\nu}{5} \int_0^s |\Delta v|^2 dt. 
\]

(3.4)

Thanks to Hölder’s inequality, Agmon’s inequality, Cauchy–Schwarz inequality, Young’s inequality and Poincaré inequality, we carry out the estimates for \(J_3\) as follows:
\[
J_3 = 8 \int_0^s |\langle (v \cdot \nabla) v, \Delta v \rangle| dt \leq 8 \int_0^s \|v\|_{L^\infty(M\gamma)^2} |\nabla v|_{L^2(M\gamma)^2} |\Delta v|_{L^2(M\gamma)^2} dt 
\]
\[ J_4 = 8 \int_0^s |\nabla \cdot (hv), \Delta h| \, dt \leq 8 \int_0^s |\langle \nabla h v, \Delta h \rangle| \, dt + 8 \int_0^s |h \nabla \cdot v, \Delta h| \, dt \]
\[ \leq C \int_0^s |\nabla h|_{L^4(M)} \| v \|_{L^4(M)^2} |\Delta h|_{L^2(M)} \, dt + C \int_0^s |h|_{L^4(M)} |\nabla v|_{L^4(M)} |\Delta h|_{L^2(M)} \, dt \]
\[ \leq C \int_0^s \left( |\nabla h|_{L^2}^{\frac{3}{2}} |\Delta h|_{L^2}^{\frac{3}{2}} \| v \|_{L^2} + |\nabla h|_{L^2} |\nabla v|_{L^2}^{\frac{3}{2}} |\Delta h|_{L^2}^{\frac{3}{2}} \right) \, dt \]
\[ \leq C \int_0^s |h|_{L^2}^{\frac{1}{2}} |\Delta h|_{L^2}^{\frac{1}{2}} \| v \| \, dt + C \int_0^s |h|_{L^2} |\nabla v|_{L^2}^{\frac{1}{2}} |\Delta h|_{L^2} \, dt \]
\[ \leq \delta \int_0^s |\Delta h|^2 \, dt + C \int_0^s (\| v \|^2 + |h|^2) \, dt + \frac{\nu}{5} \int_0^s |\Delta v|^2 \, dt. \]  

Next, thanks to Poincaré inequality, we estimate
\[ J_5 = 8 \int_0^s |\langle f k \times v, \Delta v \rangle| \, dt \leq C \int_0^s \| v(r) \|^2 + \frac{\nu}{5} \int_0^s |\Delta v|^2 \, dt. \]  

By using the Lipschitz assumption (2.9), we find
\[ J_6 + J_7 \leq 8K_V(1 + |v|^2 + |h|^2). \]

Combining (3.3)–(3.8), we obtain
\[ \sup_{t \in [0,s]} \|v(t)\|^2 + \|h(t)\|^2 + \int_0^s |\Delta v|^2 \, dt + \delta \int_0^s |\Delta h|^2 \, dt \]
\[ \leq 4(\|v(0)\|^2 + |h(0)|^2) + C \int_0^s |F|^2 \, dt + C \int_0^s [1 + |v(t)|^2 + |h(t)|^2] \, dt \]
\[ + C \int_0^s (\| v \|^2 + |h|^2)^3 + 8(J_8 + J_9) \]
\[ \leq 4(\|v(0)\|^2 + |h(0)|^2) + C \int_0^s (|F|^2 + 1) \, dt \]
where $C$ is an appropriate constant which may be different at each occurrence.

We obtain after taking expected values on both sides, and remembering that $F$ is deterministic

\[
\mathbb{E}\left( \sup_{r \in [0,s]} \|v(r)\|^2 + \sup_{r \in [0,s]} \|h(r)\|^2 + \nu \int_0^s |\Delta v|^2 \, dt + \delta \int_0^s |\Delta h|^2 \, dt \right) \leq \mathbb{E}\left( 4(\|v(0)\|^2 + \|h(0)\|^2) \right)
\]

\[
+ C \int_0^s (1 + |F|^2) \, dt + \int_0^s \sup_{0 \leq r \leq t} (\|v(r)\|^2 + \|h(r)\|^2)(\|v(r)\|^2 + \|h(r)\|^2)^2 + 1 \, dt
\]

\[
+ 8\mathbb{E}\left( \sup_{0 \leq r \leq s} \left\| \sum_{k=1}^\infty \langle \sigma_1(v, h)e_k, \Delta v \rangle dW^k_1 \right\| \right) + 8\mathbb{E}\left( \sup_{0 \leq r \leq s} \left\| \sum_{k=1}^\infty \langle \sigma_2(v, h)e_k, \Delta h \rangle dW^k_2 \right\| \right).
\]

(3.10)

We observe that since $V_1 = H_1^3(M)$, we have for $\sigma_i(v, h)e_k \in V$, $i = 1, 2$, $\Delta u \in H$,

\[
\langle \sigma_i(v, h)e_k, \Delta u \rangle = \int_M \sigma_i(v, h)e_k \cdot \Delta u \, dM.
\]

(3.11)

By integrating by parts, this is equal to

\[
- \int_M \nabla \sigma_i(v, h)e_k \cdot \nabla u \, dM + \int_M \sigma_i(v, h)e_k(\nabla u \cdot n) \, ds = - \int_M \nabla \sigma_i(v, h)e_k \cdot \nabla u \, dM.
\]

By the above expression, the Burkholder–Davis–Gundy inequality and by the Lipschitz assumptions (2.9), the two stochastic terms are bounded as follows:

\[
\mathbb{E}\left( \sup_{0 \leq r \leq s} \left\| \sum_{k=1}^\infty \langle \sigma_1(v, h)e_k, \Delta v \rangle dW^k_1 \right\| \right) \leq (\text{with } G_{e_k} = G_k = \langle \sigma_1(v, h)e_k, \Delta v \rangle)
\]

\[
\leq C \mathbb{E}\left( \int_0^s \sum_{k=1}^\infty \langle \sigma_1(v, h)e_k, \Delta v \rangle^2 \, dt \right)^{\frac{1}{2}} \leq C \mathbb{E}\left( \int_0^s \sum_{k=1}^\infty \langle \nabla \sigma_1(v, h)e_k, \nabla v \rangle^2 \, dt \right)^{\frac{1}{2}}
\]

\[
\leq C \mathbb{E}\left( \int_0^s \sum_{k=1}^\infty |\nabla \sigma_1(v, h)e_k|^2_{H_1^2} |\nabla v|^2_{H_1^2} \, dt \right)^{\frac{1}{2}} \leq C \mathbb{E}\left( \sup_{0 \leq r \leq s} \|v(r)\|^2 \int_0^s \sum_{k=1}^\infty \|\sigma_1(v, h)e_k\|^2 \, dt \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \mathbb{E}\left( \sup_{0 \leq r \leq s} \|v(r)\|^2 \right) + C \mathbb{E}\left( \int_0^s \sum_{k=1}^\infty \|\sigma_1(v, h)e_k\|^2 \, dt \right)
\]

\[
\leq \frac{1}{2} \mathbb{E}\left( \sup_{0 \leq r \leq s} \|v(r)\|^2 \right) + C \mathbb{E}\left( \int_0^s (1 + \|v(t)\|^2 + \|h(t)\|^2) \, dt \right).
\]

(3.12)

And we also find

\[
\mathbb{E}\left( \sup_{0 \leq r \leq s} \int_0^s \sum_{k=1}^\infty \langle \sigma_2(v, h)e_k, \Delta h \rangle dW^k_2 \right) \leq \frac{1}{2} \mathbb{E}\left( \sup_{0 \leq r \leq s} \|h(r)\|^2 \right) + C \mathbb{E}\left( \int_0^s (1 + \|v(t)\|^2 + \|h(t)\|^2) \, dt \right).
\]

(3.13)
Rearranging (3.9)–(3.13) and multiplying by 2, we arrive at

\[
\mathbb{E}\left(\sup_{r \in [0,s]} \|v(r)\|^2 + \sup_{r \in [0,s]} \|h(r)\|^2 + \nu \int_0^s |\Delta v|^2 \, dt + \delta \int_0^s |\Delta h|^2 \, dt \right)
\]

\[
\leq 8\mathbb{E}\left(\|v(0)\|^2 + \|h(0)\|^2\right)
\]

\[
+ C\left(\int_0^s (1 + |F|^2) \, dt\right) + C\mathbb{E}\left(\int_0^s \sup_{0 \leq r \leq t} (\|v(r)\|^2 + \|h(r)\|^2)(\|v(r)\|^2 + \|h(r)\|^2 + 1) \, dt\right)
\]

\[
+ C\mathbb{E}\left(\int_0^s (1 + \|v(r)\|^2 + \|h(r)\|^2) \, dr\right)
\]

\[
\leq 8\mathbb{E}\left(\|v(0)\|^2 + \|h(0)\|^2\right)
\]

\[
+ C\left(\int_0^s (1 + |F|^2) \, dt\right) + C\mathbb{E}\left(\int_0^s \sup_{0 \leq r \leq t} (\|v(r)\|^2 + \|h(r)\|^2)(2 + \|v(r)\|^2 + \|h(r)\|^2) \, dt\right).
\]

(3.14)

Now, we assume that \( M > 1 \) and consider the stopping time

\[
\tau = \tau_M := \inf_{s \geq 0} \left\{ \sup_{r \in [0,s]} (\|v(r)\|^2 + \|h(r)\|^2) > M \right\}
\]

(3.15)

Replacing \( s \) by \( s \wedge \tau \) in (3.14) yields

\[
\mathbb{E}\left(\sup_{r \in [0,s \wedge \tau]} \|v(r)\|^2 + \sup_{r \in [0,s \wedge \tau]} \|h(r)\|^2 + \nu \int_0^{s \wedge \tau} |\Delta v|^2 \, dt + \delta \int_0^{s \wedge \tau} |\Delta h|^2 \, dt \right)
\]

\[
\leq \mathcal{K}_1 + C(2 + M^2)\mathbb{E}\left(\int_0^{s \wedge \tau} (\|v(r)\|^2 + \|h(r)\|^2) \, dt\right),
\]

(3.16)

where

\[
\mathcal{K}_1 := 8\mathbb{E}\left(\|v(0)\|^2 + \|h(0)\|^2\right) + C\left(\int_0^s (1 + |F|^2) \, dt\right)
\]

(3.17)

Now, we define

\[
\mathcal{Y}(t) := \mathbb{E}\left(\int_0^{t \wedge \tau} (\|v(r)\|^2 + \|h(r)\|^2) \, ds\right).
\]

(3.18)

Then (3.16) implies

\[
\mathcal{Y}'(s) \leq \mathcal{K}_1 + C(2 + M^2)\mathcal{Y}(s)
\]

(3.19)
This gives
\[ \mathcal{Y}(s) \leq \frac{K_1}{C(2 + M^2)} (e^{C(2 + M^2)s} - 1). \] (3.20)

From (3.16), (3.18) and (3.20), we obtain
\[ \mathbb{E} \left( \sup_{r \in [0, s \wedge \tau]} \| v(r) \|^2 + \sup_{r \in [0, s \wedge \tau]} \| h(r) \|^2 + \nu \int_0^{s \wedge \tau} | \Delta v |^2 dt + \delta \int_0^{s \wedge \tau} | \Delta h |^2 dt \right) \leq K_1 + \frac{K_1}{C(2 + M^2)} e^{C(2 + M^2)s} C(2 + M^2) \leq K_1 + K_1 e^{C(2 + M^2)s}. \] (3.21)

The right hand side of (3.21) is bounded by \( M \) if
\[ s \leq \frac{K_1}{C(2 + M^2)} \log \frac{M - K_1}{K_1} := s_M. \]

As long as \( M \) is large enough such that \( M - K_1 > K_1 \) or \( M > 2K_1 \), we obtain that \( s_M > 0 \) and we find
\[ 0 < s \leq s_M, \]
which yields the local existence in time.

Hence
\[ \mathbb{E} \left( \sup_{r \in [0, s \wedge \tau]} \| v(r) \|^2 + \sup_{r \in [0, s \wedge \tau]} \| h(r) \|^2 + \nu \int_0^{s \wedge \tau} | \Delta v |^2 dt + \delta \int_0^{s \wedge \tau} | \Delta h |^2 dt \right) \]
\[ \leq K_1 + \frac{K_1}{C(2 + M^2)} e^{C(2 + M^2)s} C(2 + M^2) \leq K_1 + K_1 e^{C(2 + M^2)s}. \] (3.22)

is bounded by \( M \) for \( 0 < s \leq \frac{K_1}{C(2 + M^2)} \log \frac{M - K_1}{K_1} \), with \( M > 2K_1 \) where \( K_1 \) is defined at (3.17).

4. The modified system with a cut-off function

We aim to study in this section the martingale solutions of the following modified system:

\[ dv + (\nu \Delta v + \theta(\| v \|^2 + \| h \|^2) (v \cdot \nabla) v + g \nabla h + f k \times v) dt = F dt + \sum_{k=1}^{\infty} \sigma_1^k(v, h) dW_1^k, \] \[ dh + \theta(\| v \|^2 + \| h \|^2) \nabla \cdot (h v) - \delta \Delta h dt = \sum_{k=1}^{\infty} \sigma_2^k(v, h) dW_2^k, \] \[ v(0) = v_0, \quad h(0) = h_0 > 0. \] (4.1a, 4.1b, 4.1c)

Here, \( \theta : \mathbb{R} \to [0, 1] \) is a \( C^\infty \) cut-off function satisfying
\[ \theta(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq M, \\ 0 & \text{if } |\xi| \geq 2M, \end{cases} \] (4.2)

where \( M \) is any arbitrary, positive constant which is independent of \( n \). A specific appropriate choice for \( M \) will be made in the next section.
Theorem 4.1 (Global existence of martingale solutions to the modified system). With the same assumptions as in Theorem 2.1, there exists a global martingale solution to (4.1).

Theorem 4.2 (Global existence of pathwise solutions to the modified system). Under the same assumptions as in Theorem 2.2, there exists a global pathwise solution to (4.1) relative to the given stochastic basis \( S = (\Omega, \mathcal{F}, \mathbb{P}) \).

4.1. The Galerkin scheme

Considering the projection \( P_n \) defined as in (2.4), we introduce the Galerkin approximation \((v^n, h^n)\) of (4.1), with \( v^n \) and \( h^n \) functions from some interval \((0, \tau_n)\) into \( P_n(V_1 \times V_2) \), namely

\[
dv^n - \nu \Delta v^n dt + P_n[\theta((\|v^n\|^2 + \|h^n\|^2)(v^n \cdot \nabla)v^n + g \nabla h^n + f k^n \times v^n)] dt = P_nF dt + \sum_{k=1}^{\infty} P_n \sigma_1(v^n, h^n) e_k dW_1^k, \quad (4.3a)
\]

\[
dh^n + P_n[\theta((\|v^n\|^2 + \|h^n\|^2)\nabla \cdot (h^n v^n)) - \delta \Delta h^n] dt = \sum_{k=1}^{\infty} P_k \sigma_2(v^n, h^n) e_k dW_2^k, \quad (4.3b)
\]

\[
v^n(0) = v^n_0 = P_n v_0, \quad h^n(0) = h^n_0 = P_n h_0. \quad (4.3c)
\]

Recall that, by abuse of notations, we write \( P_n v = P_n(v, 0) \) and \( P_n h = P_n(0, h) \). Here \((v^n, h^n)\) are adapted processes in \( C([0, T]; H_n)^3 \sim C([0, T]; \mathbb{R}^3) \). The global existence and uniqueness of \((v^n, h^n)\) follows from the Itô Lemma, and the a priori estimates below, see [23].

4.2. Uniform estimates for the modified Galerkin system

The essential estimate for our study below is the following:

Lemma 4.1. Under the same assumptions as in Theorem 2.1 and \( \mathbb{E}\left(\|v_0\|^2 + \|h_0\|^2\right) < \infty \), we obtain the following estimates on \( v^n \) and \( h^n \)

\[
\mathbb{E}\left(\sup_{0 \leq s \leq T} \|v^n(s)\|^2 + \sup_{0 \leq s \leq T} \|h^n(s)\|^2\right) \leq \mathcal{K}_1, \quad (4.4a)
\]

and

\[
\mathbb{E}\left(\int_0^T |\Delta v^n|^2 dt + \int_0^T |\Delta h^n|^2 dt\right) \leq \mathcal{K}_2, \quad (4.4b)
\]

where \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are independent of \( n \) and depend only on the data.

Proof of Lemma 4.1. For the sake of simplicity, we denote \( \theta((\|v^n\|^2 + \|h^n\|^2)) \) by \( \theta(v^n, h^n) \). We apply the Itô formulas in (2.18) and (2.19) with \( p = 2 \) for \( v^n \) and \( h^n \) and add the corresponding relations which yields:

\[
d\|v^n\|^2 + 2\nu|\Delta v^n|^2 dt + d\|h^n\|^2 + 2\delta|\Delta h^n|^2 dt
\]

\[
= 2(P_nF, \Delta v^n)dt - 2g(\nabla h^n, \Delta v^n)dt - 2(P_n[\theta(v^n, h^n)(v^n \cdot \nabla)v^n], \Delta v^n)dt
\]

\[
- 2(P_n[\theta(v^n, h^n)\nabla \cdot (h^n v^n)], \Delta h^n)dt - (f k \times v, \Delta v^n)dt
\]
\[ + \sum_{k=1}^{\infty} ||P_n \sigma_1 (v^n, h^n)e_k||^2 dt + \sum_{k=1}^{\infty} ||P_n \sigma_2 (v^n, h^n)e_k||^2 dt \]
\[ + 2 \sum_{k=1}^{\infty} \langle P_n \sigma_1 (v^n, h^n)e_k, \Delta v^n \rangle dW_1^k + 2 \sum_{k=1}^{\infty} \langle P_n \sigma_2 (v^n, h^n)e_k, \Delta h^n \rangle dW_2^k. \]

We integrate (4.5) in time over \([0, r]\) for \(0 \leq r \leq s \leq T\), take the supremum in \(r\) over \([0, s]\); we deduce that:

\[
\sup_{0 \leq r \leq s} ||v^n(r)||^2 + 2\nu \int_0^s |\Delta v^n(t)|^2 dt + \sup_{0 \leq r \leq s} ||h^n(s)||^2 + 2\delta \int_0^s |\Delta h^n(t)|^2 dt \leq ||v^n(0)||^2 + ||h^n(0)||^2 + \int_0^s |\langle P_n F, \Delta v^n \rangle| dt + \int_0^s |\langle \nabla h^n, \Delta v^n \rangle| dt
\]
\[
+ \int_0^s |\langle P_n [\theta(v^n, h^n)(v^n \cdot \nabla) v^n], \Delta v^n \rangle| dt + \int_0^s |\langle P_n [\theta(v^n, h^n) \nabla (h^n v^n)], \Delta h^n \rangle| dt
\]
\[
+ \int_0^s |\langle f, v, \Delta v^n \rangle| dt + \int_0^s \sum_{k=1}^{\infty} ||P_n \sigma_1 (v^n, h^n)e_k||^2 dt + \int_0^s \sum_{k=1}^{\infty} ||P_n \sigma_2 (v^n, h^n)e_k||^2 dt
\]
\[
+ \sup_{0 \leq r \leq s} \left| \int_0^r \sum_{k=1}^{\infty} \langle P_n \sigma_1 (v^n, h^n)e_k, \Delta v^n \rangle dW_1^k \right| + \sup_{0 \leq r \leq s} \left| \int_0^r \sum_{k=1}^{\infty} \langle P_n \sigma_2 (v^n, h^n)e_k, \Delta h^n \rangle dW_2^k \right|
\]
\[ \leq ||v^n(0)||^2 + ||h^n(0)||^2 + J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9. \]

Note the symbol \(\leq\) means \(\leq\) up to a multiplicative constant. Also note that we can drop \(P_n\) in \(J_3, J_4, J_8, J_9\) because \(P_n\) is self-adjoint and \(P_n \Delta v^n = \Delta v^n\). We observe that due to the cut-off function, the bounds for two nonlinear deterministic terms are much more simple than (3.5) and (3.6). We proceed to treat each \(J_i\) as follows:

\[
J_1 = \int_0^s |\langle P_n F, \Delta v^n \rangle| dt \leq \frac{4}{\nu} \int_0^s |F|^2 dt + \frac{\nu}{4} \int_0^s |\Delta v^n|^2 dt. \]  

(4.7)

By Cauchy Schwarz inequality, we obtain

\[
J_2 = \int_0^s |\langle \nabla h^n, \Delta v^n \rangle| dt \leq \frac{4}{\nu} \int_0^s ||h^n||^2 dt + \frac{\nu}{4} \int_0^s |\Delta v^n|^2 dt. \]  

(4.8)

For \(J_3\) using Agmon’s inequality, Young’s inequality and \(|\theta^{\frac{1}{2}}(v^n, h^n) \nabla v^n|_{L^2(\mathcal{M})} \leq 1\), we find

\[
\int_0^s \theta(v^n, h^n)^{\frac{1}{2}} |v^n|_{L^\infty(\mathcal{M})^2} |\Delta v^n|_{L^2(\mathcal{M})^2} dt \leq C \int_0^s \theta^{\frac{1}{2}}(v^n, h^n) |v^n|^\frac{1}{2}_{L^2(\mathcal{M})^2} |\Delta v^n|^\frac{3}{2}_{L^2(\mathcal{M})} dt
\]
\[
\leq C \int_0^s \theta^{\frac{1}{2}}(v^n, h^n) ||v^n||^{\frac{1}{2}} |\Delta v^n|^\frac{3}{2}_{L^2(\mathcal{M})} dt
\]
\[ \leq \frac{\nu}{4} \int_0^s |\Delta v^n|^2 \, dt + C. \quad (4.9) \]

Similarly,

\[ J_4 \leq \delta \int_0^s |\Delta h^n|^2 + C. \quad (4.10) \]

The next term is estimated by Cauchy–Schwarz inequality and Poincaré inequality

\[ J_5 = \int_0^s |(f_k \times v, \Delta v^n)| \, dt \leq \frac{\nu}{4} \int_0^s |\Delta v^n|^2 \, dt + C \int_0^s ||v^n(t)||^2 \, dt. \quad (4.11) \]

By using Lipschitz assumptions (2.9), we obtain:

\[ J_6 + J_7 \leq 2K_V(1 + ||v^n||^2 + ||h^n||^2). \quad (4.12) \]

Combining (4.7), (4.8), (4.9), (4.10), (4.11), (4.12), we obtain:

\[ \sup_{r \in [0,s]} ||v^n(r)||^2 + \sup_{r \in [0,s]} ||h^n(r)||^2 + \nu \int_0^s |\Delta v^n|^2 \, dt + \delta \int_0^s |\Delta h^n|^2 \, dt \]

\[ \leq ||v^n(0)||^2 + ||h^n(0)||^2 + \int_0^s |F|^2 \, dt + \int_0^s (||v^n||^2 + ||h^n||^2 + 1) \, dt + J_8 + J_9. \quad (4.13) \]

We obtain after taking expectations

\[ \mathbb{E}\left( \sup_{r \in [0,s]} ||v^n(r)||^2 + \sup_{r \in [0,s]} ||h^n(r)||^2 + \nu \int_0^s |\Delta v^n|^2 \, dt + \delta \int_0^s |\Delta h^n|^2 \, dt \right) \]

\[ \leq ||v^n(0)||^2 + ||h^n(0)||^2 + \mathbb{E}\left( \int_0^s (|F|^2 + ||v^n||^2 + ||h^n||^2 + 1) \, dt \right) \]

\[ + \mathbb{E}\left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_1(v^n, h^n)e_k, \Delta v^n \rangle \, dW_1^k \right| \right) + \mathbb{E}\left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_2(v^n, h^n)e_k, \Delta h^n \rangle \, dW_2^k \right| \right). \quad (4.14) \]

The last two stochastic terms are carried out exactly as in (3.12) and (3.13). We only replace \( v \) and \( h \) by \( v^n \) and \( h^n \) and we obtain the following estimates

\[ \mathbb{E}\left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_1(v^n, h^n)e_k, \Delta v^n \rangle \, dW_1^k \right| \right) \]

\[ \leq \frac{1}{2} \mathbb{E}\left( \sup_{r \in [0,s]} ||v^n(r)||^2 \right) + C \mathbb{E}\left( \int_0^s (1 + ||v^n(t)||^2 + ||h^n(t)||^2) \, dt \right). \quad (4.15) \]
And we also find

\[
\mathbb{E}\left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k=1}^{\infty} \langle \sigma_2(\mathbf{v}^n, h^n) e_k, \Delta h^n \rangle dW^k_2 \right| \right) \\
\leq \frac{1}{2} \mathbb{E}\left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^2 + \sup_{r \in [0,s]} \|h^n(r)\|^2 \right) + CE \left( \int_0^s (1 + \|\mathbf{v}^n(t)\|^2 + \|h^n(t)\|^2) dt \right).
\]

(4.16)

Rearranging (4.7) to (4.16), we arrive at

\[
\mathbb{E}\left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^2 + \sup_{r \in [0,s]} \|h^n(r)\|^2 \right) + \nu \int_0^s \|\Delta \mathbf{v}^n(t)\|^2 dt + \delta \int_0^s \|\Delta h^n(t)\|^2 dt \\
\leq E \left( \|\mathbf{v}^n(0)\|^2 + \|h^n(0)\|^2 \right) + \left( \int_0^s |F|^2 dt \right) + \mathbb{E}\left( \int_0^s (\|\mathbf{v}^n\|^2 + \|h^n\|^2) dt \right).
\]

(4.17)

Since \( \mathbb{E}\left( \int_0^s (\|\mathbf{v}^n\|^2 + \|h^n\|^2) dt \right) \leq \int_0^s \mathbb{E} \sup_{0 \leq r \leq t} (\|\mathbf{v}^n(r)\|^2 + \|h^n(r)\|^2) dt \), we obtain

\[
\mathbb{E}\left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^2 + \sup_{r \in [0,s]} \|h^n(r)\|^2 \right) + \nu \int_0^s \|\Delta \mathbf{v}^n(t)\|^2 dt + \delta \int_0^s \|\Delta h^n(t)\|^2 dt \\
\leq E \left( \|\mathbf{v}^n(0)\|^2 + \|h^n(0)\|^2 \right) + \left( \int_0^s |F|^2 dt \right) + \mathbb{E}\left( \sup_{0 \leq r \leq t} (\|\mathbf{v}^n(r)\|^2 + \|h^n(r)\|^2) dt \right).
\]

(4.18)

By applying the deterministic Gronwall inequality to \( Y(s) = E \left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^2 + \sup_{r \in [0,s]} \|h^n(r)\|^2 \right) \), we obtain:

\[
\mathbb{E}\left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^2 + \sup_{r \in [0,s]} \|h^n(r)\|^2 \right) \leq E \left( \|\mathbf{v}^n(0)\|^2 + \|h^n(0)\|^2 \right) + \int_0^T |F|^2 dt \\
\leq E \left( \|\mathbf{v}(0)\|^2 + \|h(0)\|^2 \right) + \int_0^T |F|^2 dt.
\]

(4.19)

The proof of the lemma will be complete once we prove that

\[
\mathbb{E}\left( \int_0^T |\Delta \mathbf{v}^n|^2 dt + \int_0^T |\Delta h^n|^2 dt \right) \leq K_2.
\]

(4.20)

From (4.14) (4.15) and (4.16), we infer that:

\[
\mathbb{E}\left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^2 + \sup_{r \in [0,s]} \|h^n(r)\|^2 + \nu \int_0^s |\Delta \mathbf{v}^n|^2 dt + \delta \int_0^s |\Delta h^n|^2 dt \right)
\]
We can extend the results of the Lemma 4.1 as follows. We will see that the extension is needed to establish the compactness result which is the crucial point to obtain the existence of the martingale solutions.

**Lemma 4.2.** If under the same assumptions as in Theorem 2.1 we additionally suppose that $p > 2$, $F \in L^p(0, T; H_1)$ and $\mathbb{E}\left(\|v^n(0)\|^p + \|h^n(0)\|^p\right) < \infty$; then we have:

\[
\mathbb{E}\left(\sup_{0 \leq s \leq T} \|v^n(s)\|^p + \sup_{0 \leq s \leq T} \|h^n(s)\|^p\right) \leq K_3, \tag{4.21}
\]

where $K_3$ is independent of $n$ and depends only on the data.

**Proof of Lemma 4.2.** Recall (4.13):

\[
\sup_{r \in [0, s]} \|v^n(r)\|^p + \sup_{r \in [0, s]} \|h^n(r)\|^p \leq \|v^n(0)\|^p + \|h^n(0)\|^p + \int_0^s \|F\|^p dt + \int_0^s \|\Delta v^n\|^p dt + \int_0^s \|\Delta h^n\|^p dt
\]

\[
+ \sup_{r \in [0, s]} \left| \sum_{k=1}^{\infty} \langle \sigma_1(v^n, h^n)e_k, \Delta v^n \rangle dW^k_1 \right| + \sup_{r \in [0, s]} \left| \sum_{k=1}^{\infty} \langle \sigma_2(v^n, h^n)e_k, \Delta h^n \rangle dW^k_2 \right|
\]

Dropping the two integrals on the LHS of the above inequality and raising both sides to the power $p/2$, $p > 2$, we obtain thanks to the Minkowski inequality

\[
\sup_{r \in [0, s]} \|v^n(r)\|^p + \sup_{r \in [0, s]} \|h^n(r)\|^p \leq \|v^n(0)\|^p + \|h^n(0)\|^p + \int_0^s \|F\|^p dt + \int_0^s \|\Delta v^n\|^p dt + \int_0^s \|\Delta h^n\|^p dt
\]

\[
+ \sup_{r \in [0, s]} \left| \sum_{k=1}^{\infty} \langle \sigma_1(v^n, h^n)e_k, \Delta v^n \rangle dW^k_1 \right| + \sup_{r \in [0, s]} \left| \sum_{k=1}^{\infty} \langle \sigma_2(v^n, h^n)e_k, \Delta h^n \rangle dW^k_2 \right|
\]

We take the expected values on both sides and obtain:

\[
\mathbb{E}\left(\sup_{r \in [0, s]} \|v^n(r)\|^p + \sup_{r \in [0, s]} \|h^n(r)\|^p\right)
\]

\[
\leq \mathbb{E}\left(\|v^n(0)\|^p + \|h^n(0)\|^p\right) + \left( \int_0^s \|F\|^p dt \right) + \mathbb{E}\left( \int_0^s \|\Delta v^n\|^p dt + \int_0^s \|\Delta h^n\|^p dt \right)
\]

\[
+ \mathbb{E}\left( \sup_{r \in [0, s]} \left| \sum_{k=1}^{\infty} \langle \sigma_1(v^n, h^n)e_k, \Delta v^n \rangle dW^k_1 \right| \right)
\]
\[ + \mathbb{E} \left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_2(\mathbf{v}^n, h^n) e_k, \Delta h^n \rangle dW^k_2 \right|^{\frac{q}{2}} \right). \tag{4.22} \]

By applying the BDG inequality \((2.12)\) with \(r = \frac{p}{2} > 1\), the two stochastic terms on the RHS of \((4.22)\) are estimated as follows:

\[ \mathbb{E} \left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma^k(\mathbf{v}^n, h^n) e_k, \Delta \mathbf{v}^n \rangle dW^k_1 \right|^{\frac{q}{2}} \right) \leq C_1 \mathbb{E} \left( \int_0^s \|\mathbf{v}^n\|^2 \sum_{k=1}^\infty \|\sigma_1(\mathbf{v}^n, h^n) e_k\|^2 dt \right)^{\frac{q}{4}} \leq C_1 \mathbb{E} \left( \int_0^s \sup_{r \in [0,s]} (\|\mathbf{v}^n\|^2 + \|\mathbf{h}^n\|^2) dt \right)^{\frac{p}{4}} \]

\[ = C_1 \mathbb{E} \left( \int_0^s \|\mathbf{v}^n\|^2 \|\sigma_1(\mathbf{v}^n, h^n)\|^2_{L^2(U_{\mathcal{V}_1})} dt \right)^{\frac{p}{4}} \leq C_1 \mathbb{E} \left( \int_0^s \sup_{r \in [0,s]} (\|\mathbf{v}^n\|^2 + \|\mathbf{h}^n\|^2) dt \right)^{\frac{p}{4}} \]

\[ \leq \frac{1}{2} \mathbb{E} \sup_{r \in [0,s]} \|\mathbf{v}^n\|^p + CE \left( \int_0^s (1 + \|\mathbf{v}^n\|^p + \|\mathbf{h}^n\|^p) dt \right). \tag{4.23} \]

Similarly

\[ \mathbb{E} \left( \sup_{r \in [0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_2(\mathbf{v}^n, h^n) e_k, \Delta h^n \rangle dW^k_2 \right|^{\frac{q}{2}} \right) \leq \frac{1}{2} \mathbb{E} \sup_{r \in [0,s]} \|\mathbf{h}^n\|^p + 2C \mathbb{E} \left( \int_0^s (1 + \|\mathbf{v}^n\|^p + \|\mathbf{h}^n\|^p) dt \right). \tag{4.24} \]

Rearranging \((4.22), (4.23)\) and \((4.24)\) and multiplying by 2, we obtain

\[ \mathbb{E} \left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^p + \sup_{r \in [0,s]} \|\mathbf{h}^n(r)\|^p \right) \]

\[ \leq \mathbb{E} (\|\mathbf{v}^n(0)\|^p + \|\mathbf{h}^n(0)\|^p) + \left( \int_0^s \|F\|^p \ dt \right) + \mathbb{E} \left( \int_0^s (\|\mathbf{v}^n\|^p + \|\mathbf{h}^n\|^p + 1) dt \right). \]

Since \( \mathbb{E} \left( \int_0^s (\|\mathbf{v}^n\|^p + \|\mathbf{h}^n\|^p) dt \right) \leq \mathbb{E} \left( \int_0^s \sup_{r \in [0,t]} (\|\mathbf{v}^n(r)\|^p + \sup_{r \in [0,t]} \|\mathbf{h}^n(r)\|^p) dt \right) \), we have

\[ \mathbb{E} \left( \sup_{r \in [0,s]} \|\mathbf{v}^n(r)\|^p + \sup_{r \in [0,s]} \|\mathbf{h}^n(r)\|^p \right) \leq \mathbb{E} (\|\mathbf{v}^n(0)\|^p + \|\mathbf{h}^n(0)\|^p) + \left( \int_0^s \|F\|^p \ dt \right) \]

\[ + \mathbb{E} \left( \int_0^s \left( \sup_{r \in [0,t]} (\|\mathbf{v}^n(r)\|^p + \sup_{r \in [0,t]} \|\mathbf{h}^n(r)\|^p + 1) dt \right) \right). \]
By applying the deterministic Gronwall inequality to
\[ Y(s) = \mathbb{E} \left( \sup_{r \in [0,s]} \| v^n(r) \|^p + \sup_{r \in [0,s]} \| h^n(r) \|^p \right), \]
we arrive at
\[ \mathbb{E} \left( \sup_{r \in [0,s]} \| v^n(r) \|^p + \sup_{r \in [0,s]} \| h^n(r) \|^p \right) \leq (\| v^n(0) \|^p + \| h^n(0) \|^p) + \left( \int_0^s |F|^p \, dt \right), \]
\[ \leq (\| v(0) \|^p + \| h(0) \|^p) + \left( \int_0^s |F|^p \, dt \right). \] (4.25)

The lemma is now proved. \( \square \)

**Lemma 4.3 (Estimates in fractional Sobolev spaces).** Under the same assumptions as in Theorem 2.1, we consider the associated sequence of solutions \( \{ (v^n, h^n) \}_{n \geq 1} \) of the Galerkin system (4.1). Let \( p > 2 \) and \( \mathbb{E}(|v_0|^2 + |h_0|^2) < \infty \); then there exists a finite number \( K > 0 \) (depending only on the data) such that

\[ \mathbb{E} \left( \left| t \sum_{k=1}^{\infty} P_n \sigma^k_1(v^n, h^n)dW_1^k \right|^p \right) \leq K, \] (4.26a)
\[ \mathbb{E} \left( \left| t \sum_{k=1}^{\infty} P_n \sigma^k_2(v^n, h^n)dW_2^k \right|^p \right) \leq K, \] (4.26b)
\[ \mathbb{E} \left( \left| v^n(t) - \int_0^t \sum_{k=1}^{\infty} P_n \sigma^k_1(v^n, h^n)dW_1^k \right|^2 \right) \leq K, \] (4.26c)
\[ \mathbb{E} \left( \left| h^n(t) - \int_0^t \sum_{k=1}^{\infty} P_n \sigma^k_2(v^n, h^n)dW_2^k \right|^2 \right) \leq K. \] (4.26d)

**Proof of Lemma 4.3.** We frequently make use of (2.13)

\[ \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t \sum_{k=1}^{\infty} G_e dW^k \right|^p \right) \leq C \mathbb{E} \left( \int_0^T \sum_{k=1}^{\infty} |G_e|^p \, dt \right). \] \( \square \)

**Proof of (4.26a).** We have for \( G_e = \sigma_1(v^n, h^n)e_k = \sigma_1(v^n, h^n)e_k \)

\[ \mathbb{E} \left( \left| \int_0^t \sum_{k=1}^{\infty} P_n \sigma_1(v^n, h^n)e_k dW_1^k \right|^p \right) \leq \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t \sum_{k=1}^{\infty} P_n \sigma_1(v^n, h^n)e_k dW_1^k \right|^p \right) \]
\[
\leq c\mathbb{E} \left( \sum_{k=1}^{T} \sigma_1(v^n, h^n)\varepsilon_k^{p_{H_1}} t \right) \leq c\mathbb{E} \left( \int_{0}^{T} \left\| \sigma_1(v^n, h^n) \right\|^p_{L_2(U_1, H_1)} dt \right)
\]
\[
\leq c\mathbb{E} \left( \int_{0}^{T} (1 + |v^n|^2 + |h^n|^2) dt \right) \leq c\mathbb{E} \left( \int_{0}^{T} (1 + |v^n|^p + |h^n|^p) dt \right)
\]
\[
\leq c\mathbb{E} \left( \int_{0}^{T} (1 + \|v^n\|^p + \|h^n\|^p) dt \right) \leq c\mathbb{E} \left( \int_{0}^{T} (1 + \sup_{t \in [0,T]} \|v^n\|^p + \sup_{t \in [0,T]} \|h^n\|^p) dt \right)
\]
\[
\leq K \text{ (from (4.25)). } \square
\]

The proof of (4.26b) follows analogously.

**Proof of (4.26c).** Integrating (4.3a) on \((0, t), 0 \leq t \leq T\), we obtain

\[
v^n - \nu \int_{0}^{t} \Delta v^n ds + \int_{0}^{t} P_n[\theta(v^n, h^n)(v^n \cdot \nabla)v^n + g\nabla h^n + f k \times v^n] ds
\]
\[
= \int_{0}^{t} P_n F ds + \int_{0}^{t} \sum_{k=1}^{\infty} P_n \sigma_1^k(v^n, h^n) dW_1^k + v_0.
\]

Therefore

\[
v^n - \int_{0}^{t} \sum_{k=1}^{\infty} P_n \sigma_1^k(v^n, h^n) dW_1^k
\]
\[
= \nu \int_{0}^{t} \Delta v^n ds + \int_{0}^{t} P_n F ds - \int_{0}^{t} P_n[\theta(v^n, h^n)(v^n \cdot \nabla)v^n + g\nabla h^n + f k \times v^n] ds + v_0.
\]

Taking expectations on both sides of the above expression, we obtain:

\[
\mathbb{E} \left( \left| v^n - \int_{0}^{t} \sum_{k=1}^{\infty} P_n \sigma_1(v^n, h^n)\varepsilon_k dW_1^k \right|^2 \right)
\]
\[
= \mathbb{E} \left( \int_{0}^{T} \left| v^n - \int_{0}^{t} \sum_{k=1}^{\infty} P_n \sigma_1(v^n, h^n)\varepsilon_k dW_1^k \right|^2_{H_1} \right) + \mathbb{E} \left( \int_{0}^{T} \frac{d}{dt} \left| v^n - \int_{0}^{t} \sum_{k=1}^{\infty} P_n \sigma_1(v^n, h^n)\varepsilon_k dW_1^k \right|^2_{H_1} \right)
\]
\[
= \mathbb{E} \left( \int_{0}^{T} |v_0|^2 + |\nu \Delta v^n + P_n[\theta(v^n, h^n)(v^n \cdot \nabla)v^n + g\nabla h^n + f k \times v^n]|^2_{H_1} dt + \int_{0}^{T} |F|^2 dt \right)
\]
\[
\leq \mathbb{E} \left( |v_0|^2 + \nu \int_{0}^{T} |\Delta v^n|^2 dt + \int_{0}^{T} |\nabla v^n|^2 dt + g \int_{0}^{T} |\nabla h^n| dt + \int_{0}^{T} |v^n|^2 dt + \int_{0}^{T} |F|^2 dt \right)
\]
\[ \leq \mathbb{E} \left( |\mathbf{v}_0|^2 + \int_0^T |\Delta \mathbf{v}^n|^2 dt + \int_0^T \sup_{r \in [0,t]} \| \mathbf{v}^n \|^2 dt + \int_0^T \sup_{s \in [0,t]} \| h^n \|^2 dt + \int_0^T \| n \|^2 dt + \int_0^T |F|^2 dt \right) \]

The last line holds true in virtue of Lemma 4.1.

Similarly, we give the proof of (4.26d): integrating (3.1b) on [0, t], we see that

\[ h^n + \int_0^t P_n[\theta(\mathbf{v}^n, h^n) \nabla (h^n \mathbf{v}^n) - \delta \Delta h^n] dt = \int_0^t \sum_{k=1}^\infty P_n \sigma_2(\mathbf{v}^n, h^n) e_k dW_2^k + h_0. \]

Thus

\[ \mathbb{E} \left( \left| h^n - \int_0^t \sum_{k=1}^\infty P_n \sigma_2(\mathbf{v}^n, h^n) e_k dW_2^k \right|^2 \right) \]

\[ = \mathbb{E} \left( \left| \int_0^t P_n[\theta(\mathbf{v}^n, h^n) \nabla (h^n \mathbf{v}^n) - \delta \Delta h^n] dt + h_0 \right|^2 \right) \]

\[ \leq \mathbb{E} \left( |h_0|^2 + \int_0^t |P_n[\theta(\mathbf{v}^n, h^n) \nabla (h^n \mathbf{v}^n) - \delta \Delta h^n]|^2 dt \right) \]

\[ \leq \mathbb{E} \left( |h_0|^2 + \int_0^t |\nabla h^n|^2 + |\nabla \mathbf{v}^n|^2| dt + \delta \int_0^t |\Delta h^n|^2 dt \right) \]

\[ \leq \mathbb{E} \left( |h_0|^2 + \int_0^t \sup_{r \in [0,t]} \| h^n \|^2 dt + \int_0^t \sup_{r \in [0,t]} \| \mathbf{v}^n \|^2 dt + \delta \int_0^t |\Delta h^n|^2 dt \right) \]

from where, using Lemma 4.1, the result follows. \qed

4.3. Compactness arguments

For a given initial distribution \( \mu_0 \) on \( H_1 \times H_2 \), we fix a stochastic basis, \( S = (\Omega, \mathcal{F}, (\mathcal{F})_{\tau \geq 0}, \mathbb{P}, W_1, W_2) \), upon which is defined an \( \mathcal{F}_0 \)-measurable random variable \((\mathbf{v}_0, h_0)\) whose distribution is \( \mu_0 \). Then we go back to the finite dimensional approximations relative to \( S \) and \((\mathbf{v}_0, h_0)\). We define the phase space \( \mathcal{X} = \mathcal{X}_\nu \times \mathcal{X}_h \times \mathcal{X}_{W_1} \times \mathcal{X}_{W_2} \), where

\[ \mathcal{X}_\nu = L^2(0, T; V_1) \cap \mathcal{C}([0, T]; V_1'), \quad \mathcal{X}_h = L^2(0, T; V_2) \cap \mathcal{C}([0, T]; V_2'), \quad \text{and} \quad \mathcal{X}_{W_i} = \mathcal{C}([0, T]; \mathcal{U}_0). \quad (4.27) \]

We consider the probability measures

\[ \mu_\nu^n(\cdot) = \mathbb{P}(\mathbf{v}^n \in \cdot) \in Pr(L^2(0, T; V_1) \cap \mathcal{C}([0, T]; V_1')), \quad (4.28) \]

\[ \mu_h^n(\cdot) = \mathbb{P}(h^n \in \cdot) \in Pr(L^2(0, T; V_2) \cap \mathcal{C}([0, T]; V_2')), \quad (4.29) \]

and

\[ \mu_{W_i}(\cdot) = \mu_{W_i}^\nu(\cdot) = \mathbb{P}(W_i \in \cdot) \in Pr(\mathcal{C}([0, T]; \mathcal{U}_0)). \quad (4.30) \]

This defines a sequence of probability measures \( \mu^n = \mu_\nu^n \times \mu_h^n \times \mu_{W_1}^\nu \times \mu_{W_2}^\nu \) on \( \mathcal{X} \). Then we have the following tightness result:
Lemma 4.4. Consider the measures $\mu^n$ on $X$ defined as above in (4.28)–(4.30). Then the sequence $\{\mu^n\}_{n \geq 1}$ is tight and therefore weakly compact in the phase space $X$.

Proof. We apply Lemma 6.1 below with $\mathcal{E}_0 = D(-\Delta)$, $\mathcal{E}_1 = H_1$, $p = 2$, $\alpha = 1/3$, we deduce

$$L^2(0, T; D(-\Delta)) \cap W^{1/3, 2}(0, T; H_1) \subset \subset L^2(0, T; V_1),$$

and similarly

$$L^2(0, T; D(-\Delta)) \cap W^{1/3, 2}(0, T; H_2) \subset \subset L^2(0, T; V_2).$$

For $R > 0$, we define the sets

$$B^1_R = \left\{ U \in L^2(0, T; D(-\Delta)) \cap W^{1/3, 2}(0, T; H_1) : |U|^2_{L^2(0, T; D(-\Delta))} + |U|^2_{W^{1/3, 2}(0, T; H_1)} \leq R^2 \right\},$$

(4.31)

and

$$B^2_R = \left\{ U \in L^2(0, T; D(-\Delta)) \cap W^{1/3, 2}(0, T; H_2) : |U|^2_{L^2(0, T; D(-\Delta))} + |U|^2_{W^{1/3, 2}(0, T; H_2)} \leq R^2 \right\}.$$

(4.32)

From the above, we know that these sets are compact in $L^2(0, T; V_1)$ and $L^2(0, T; V_2)$. Thanks to the Chebyshev inequality, the uniform estimates (4.26), and Lemma 4.1, we obtain:

$$\mu_v^n((B^1_R)^C) = \mathbb{P}\left( \|v^n\|^2_{L^2(0, T; V_1)} + \|v^n\|^2_{W^{1/3, 2}(0, T; H_1)} \geq R^2 \right)$$

$$\leq \mathbb{P}\left( \|v^n\|^2_{L^2(0, T; V_1)} \geq \frac{R^2}{2} \right) + \mathbb{P}\left( \|v^n\|^2_{W^{1/3, 2}(0, T; H_1)} \geq \frac{R^2}{2} \right)$$

$$\leq \frac{2}{R^2} \mathbb{E}\left( \|v^n\|^2_{L^2(0, T; V_1)} + \|v^n\|^2_{W^{1/3, 2}(0, T; H_1)} \right)$$

$$\leq \frac{K}{R^2},$$

(4.33)

and

$$\mu_h^n((B^2_R)^C) = \mathbb{P}\left( \|h^n\|^2_{L^2(0, T; V_2)} + \|h^n\|^2_{W^{1/3, 2}(0, T; H_2)} \geq R^2 \right)$$

$$\leq \mathbb{P}\left( \|h^n\|^2_{L^2(0, T; V_2)} \geq \frac{R^2}{2} \right) + \mathbb{P}\left( \|h^n\|^2_{W^{1/3, 2}(0, T; H_2)} \geq \frac{R^2}{2} \right)$$

$$\leq \frac{2}{R^2} \mathbb{E}\left( \|h^n\|^2_{L^2(0, T; V_2)} + \|h^n\|^2_{W^{1/3, 2}(0, T; H_2)} \right)$$

$$\leq \frac{K}{R^2}.$$  

(4.34)

For $p > 2$ we choose $\alpha$ such that $\alpha p > 1$. We also infer further the compact embedding as in Lemma 6.2 with $\mathcal{E}_0 = H_i$, $\mathcal{E} = V_i'$, $i = 1, 2$

$$W^{1, 2}(0, T; H_i) \subset \subset C([0, T]; V_i'), \quad \text{and} \quad W^{\alpha, p}(0, T; H_i) \subset \subset C([0, T]; V_i') \text{ for } i = 1, 2.$$

Let $B^3_R$ and $B^4_R$ be the balls of radius $R$ in $W^{1, 2}(0, T; H_1)$ and $W^{\alpha, p}(0, T; H_1)$, respectively. It follows that

$$B^3_R = B^3_{R1} + B^3_{R2} \text{ is compact in } C([0, T]; V_i').$$

Similarly, let $B^4_R$ and $B^4_R$ be the balls of radius $R$ in $W^{1, 2}(0, T; H_2)$ and $W^{\alpha, p}(0, T; H_2)$, respectively. It also follows that

$$B^4_R = B^4_{R1} + B^4_{R2} \text{ is compact in } C([0, T]; (V_2)').$$
Observe that
\[
\{(v^n) \subset B^3_R \} \supset \left\{ v^n - \int_0^t \sum_k P_n \sigma_1(v^n, h^n) e_k dW^k_1 \in B^{3,1}_{R_0} \right\} \cap \left\{ \int_0^t \sum_k P_n \sigma_1(v^n, h^n) e_k dW^k_2 \in B^{3,2}_{R_0} \right\}
\)

and
\[
\{(h^n) \subset B^4_R \} \supset \left\{ h^n - \int_0^t \sum_k P_n \sigma_2(v^n, h^n) e_k dW^k_1 \in B^{4,1}_{R_0} \right\} \cap \left\{ \int_0^t \sum_k P_n \sigma_2(v^n, h^n) e_k dW^k_2 \in B^{4,2}_{R_0} \right\}
\]

By the Chebyshev inequality, the uniform bound, and the uniform estimates in (4.26), we infer
\[
\mu^n_v((B^3_R)^C) \leq \mathbb{P} \left( \left\| v^n - \int_0^t \sum_{k=1}^\infty P_n \sigma_2(v^n, h^n) e_k dW^k_1 \right\|_{W^{1,2}(0,T;H_1)}^2 \geq R^2 \right) + \mathbb{P} \left( \left\| \int_0^t \sum_{k=1}^\infty P_n \sigma_2(v^n, h^n) e_k dW^k_2 \right\|_{W^{0,p}(0,T;H_2)}^p \geq R^p \right) \leq \frac{\mathcal{K}}{R^2},
\]

(4.35)

and
\[
\mu^n_h((B^4_R)^C) \leq \mathbb{P} \left( \left\| h^n - \int_0^t \sum_{k=1}^\infty P_n \sigma_2(v^n, h^n) e_k dW^k_1 \right\|_{W^{1,2}(0,T;H_1)}^2 \geq R^2 \right) + \mathbb{P} \left( \left\| \int_0^t \sum_{k=1}^\infty P_n \sigma_2(v^n, h^n) e_k dW^k_2 \right\|_{W^{0,p}(0,T;H_2)}^p \geq R^p \right) \leq \frac{\mathcal{K}}{R^2}.
\]

(4.36)

Since \( B^1_R \cap B^3_R \) is compact in \( L^2(0,T;V_1) \cap \mathcal{C}([0,T];V_1^\prime) \) and \( B^3_R \cap B^4_R \) is compact in \( L^2(0,T;V_2) \cap \mathcal{C}([0,T];V_2^\prime) \), we have for every \( R > 0 \):
\[
\mu^n_v((B^1_R \cap B^3_R)^C) \leq \mu^n_v((B^1_R)^C) + \mu^n_v((B^3_R)^C) \leq \frac{\mathcal{K}}{R^2},
\]

(4.37)

and
\[
\mu^n_h((B^3_R \cap B^4_R)^C) \leq \mu^n_h((B^3_R)^C) + \mu^n_h((B^4_R)^C) \leq \frac{\mathcal{K}}{R^2}.
\]

(4.38)

Thus, we take \( A^c_\epsilon := B^1_{\sqrt{4K/\epsilon}} \cap B^3_{\sqrt{4K/\epsilon}} \) and \( A^h_\epsilon := B^2_{\sqrt{4K/\epsilon}} \cap B^4_{\sqrt{4K/\epsilon}} \). This gives us for any \( \epsilon > 0 \) and for every \( n \),
\[
\mu^n_v(A^c_\epsilon) \geq 1 - \frac{\epsilon}{4} \quad \text{and} \quad \mu^n_h(A^h_\epsilon) \geq 1 - \frac{\epsilon}{4}.
\]

(4.39)

For the constant sequences \( \{\mu^n_v\} \), which are weakly compact, we see that they are also tight by Proposi-
tion 6.1, and hence there exist compact sets \( \tilde{A}^i_\epsilon \subset \mathcal{C}([0,T];\mathcal{U}_0) \) such that for each \( n \),
\[
\mu^n_{W_{ij}}(\tilde{A}_{ij}) \geq 1 - \frac{\epsilon}{4}. \tag{4.40}
\]

Now, for any \( \epsilon > 0 \), we define \( \mathcal{A}_\epsilon := A^\epsilon \times A^\epsilon \times \tilde{A}_1^\epsilon \times \tilde{A}_2^\epsilon \) which are compact in \( \mathcal{X} \). We see that for every \( n \),
\[
\mu^n(\mathcal{A}_\epsilon) \geq 1 - \epsilon. \tag{4.41}
\]

This proves that the sequence \( \{\mu^n\} \) is tight in \( \mathcal{X} \) which implies by Proposition 6.1 that \( \mu^n \) is weakly compact on \( \mathcal{X} \). \( \square \)

By Lemma 4.4, we have shown that the sequence of measures \( \{\mu^n\}_{n \geq 1} \) associated with the Galerkin sequence \( \{\nu^n, h^n, W_1, W_2\} \) are weakly compact. This implies the existence of convergent subsequence \( \mu^{n_j} \) and to simplify the writing, we write \( j \) for \( n_j \).

4.4. Passage to the limit

In this section, we prove the details of the passage to the limit, which is used in the proof of both martingale solutions and pathwise solutions.

**Theorem 4.3.** Given a stochastic basis

\[
\mathcal{S} = \{(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k_t\}_{k \geq 1})\}, \quad k = 1, 2,
\]
and \( \mu_0 \) is a probability measure on \( H_1 \times H_2 \) with
\[
\int \frac{|x|^p}{d\mu_0(x)} < \infty \quad \text{for some} \quad p > 2. \quad \text{Then there exists a probability space} \quad (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{\mathbb{P}}, \{\tilde{W}^k_t\}_{k \geq 1}), \quad i = 1, 2 \quad \text{and a sequence of} \quad \mathcal{X}\text{-valued random variables} \quad (\tilde{\nu}^i, \tilde{h}^i, \tilde{W}^1_t, \tilde{W}^2_t), \quad \text{such that}
\]

(i) \( (\tilde{\nu}^i, \tilde{h}^i, \tilde{W}^1_t, \tilde{W}^2_t) \) have the same law as \( (\nu^i, h^i, W_1, W_2) \).

(ii) \( (\tilde{\nu}^i, \tilde{h}^i, \tilde{W}^1_t, \tilde{W}^2_t) \) converges almost surely in the topology of \( \mathcal{X} \) to an element \( (\tilde{\nu}, \tilde{h}, \tilde{W}_T, \tilde{W}_T) \) i.e.

\[
\tilde{\nu}^i \rightarrow \tilde{\nu} \quad \text{in} \quad L^2(0, T; V_1) \cap \mathcal{C}([0, T]; H_1) \quad \text{a.s.}, \tag{4.42a}
\]

\[
\tilde{h}^i \rightarrow \tilde{h} \quad \text{in} \quad L^2(0, T; V_2) \cap \mathcal{C}([0, T]; H_2) \quad \text{a.s.}, \tag{4.42b}
\]

\[
\tilde{W}^i_t \rightarrow \tilde{W}_t \quad \text{in} \quad \mathcal{C}([0, T]; \mathcal{U}_0), \quad i = 1, 2 \quad \text{a.s.} \tag{4.42c}
\]

(iii) Each \( \tilde{W}^i_t, \quad i = 1, 2 \) is a cylindrical Wiener process relative to the filtration \( \tilde{\mathcal{F}}^i_t \) given by

\[
\tilde{\mathcal{F}}^i_t := \sigma(\tilde{W}^i_1(s), \tilde{W}^i_2(s), \tilde{\nu}^i(s), \tilde{h}^i(s), s \leq t).
\]

(iv) Each quadruplet \( (\tilde{\nu}^i, \tilde{h}^i, \tilde{W}^1_t, \tilde{W}^2_t) \) satisfies \( \tilde{\mathbb{P}} \)-a.s.

\[
d\tilde{\nu}^i - \nu \Delta \tilde{\nu}^i dt + P_n[\theta (||\tilde{\nu}^i||^2 + ||\tilde{h}^i||^2)(\tilde{\nu}^i \cdot \nabla)\tilde{\nu}^i + g\nabla \tilde{h}^i + f\kappa \times \tilde{\nu}^i] dt \\
= F^i dt + \sum_{k=1}^{\infty} P_n (\sigma_{i,j}^k(\tilde{\nu}^i, \tilde{h}^i)e_k d\tilde{W}_1^{i,j,k}, \tag{4.43}
\]

\[
d\tilde{h}^i + P_n[\theta (||\tilde{\nu}^i||^2 + ||\tilde{h}^i||^2) \nabla \cdot (\tilde{h}^i \tilde{\nu}^i) - \delta \Delta \tilde{h}^i] dt = \sum_{k=1}^{\infty} P_n (\sigma_{2,j}^k(\tilde{\nu}^i, \tilde{h}^i) d\tilde{W}_2^{i,j,k},
\]
\[ \tilde{v}^j(0) = P_j(v_0) = v_0^j, \]
\[ \tilde{h}^j(0) = P_j(h_0) = h_0^j. \]  

Let \( \tilde{S} = (\tilde{\Omega}, \tilde{F}, (\tilde{F}_t)_{t \geq 0}, \tilde{W}_1, \tilde{W}_2) \), where
\[ \tilde{F}_t = \sigma(\tilde{W}_1(s), \tilde{W}_2(s), \tilde{v}(s), \tilde{h}(s), s \leq t). \]

Then \((\tilde{S}, \tilde{v}, \tilde{h})\) is a global martingale solution of (4.1) in the sense of Definition 2.2.

**Proof.** The proofs of (i) and (ii) can be deduced from the Skorohod embedding theorem. In order to prove (iii), it suffices to show that

(iii)\(_A\): \( \tilde{W}^j(t) \) is measurable with \( \tilde{F}_t^j \).

(iii)\(_B\): \( \tilde{W}^j(t) - \tilde{W}^j(s) \) is independent of \( \tilde{F}_t^j \).

The proofs for both (iii)\(_A\) and (iii)\(_B\) are the consequences of (i) and the facts that:

(iii)\(_C\): \( W^j(t) \) is measurable with \( F_t^j \).

(iii)\(_D\): \( W^j(t) - W^j(s) \) is independent of \( F_t^j \).

To show (iv), we refer the readers to the technique of modification as in [1].

(iv) It is easy to see that all the statistical estimates for \( v^n \) and \( h^n \) are valid for \( \tilde{v}^j \) and \( \tilde{h}^j \). Hence \((\tilde{v}^j)\) belongs to a bounded set of \( L^2(\tilde{\Omega}; L^\infty(0, T; V_1)) \cap L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))) \), and there is a \( \tilde{v} \) in this intersection space such that
\[ \tilde{v}^j \rightharpoonup \tilde{v} \text{ weak-star in } L^2(\tilde{\Omega}; L^\infty(0, T; V_1)), \]

and
\[ \tilde{v}^j \rightharpoonup \tilde{v} \text{ weakly in } L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))). \]

Similarly, there exists an \( \tilde{h} \) in \( L^2(\tilde{\Omega}; L^\infty(0, T; V_2)) \cap L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))) \) such that
\[ \tilde{h}^j \rightharpoonup \tilde{h} \text{ weak-star in } L^2(\tilde{\Omega}; L^\infty(0, T; V_2)), \]

weakly and
\[ \tilde{h}^j \rightharpoonup \tilde{h} \text{ in } L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))). \]

Our task now is to show that \( \tilde{v}, \tilde{h} \) are solutions of (4.3).

Due to Lemma 4.2 for \( p = 4 \), we obtain the following:

\[ \sup_j \mathbb{E} \left( \int_0^T \| \tilde{v}^j \|^2 \, dt \right)^2 \leq \sup_j c \mathbb{E} \left( \sup_j \| \tilde{v}^j \|^4 \right) < \infty, \]  

and
\[ \sup_j \mathbb{E} \left( \int_0^T \| \tilde{h}^j \|^2 \, dt \right)^2 \leq \sup_j c \mathbb{E} \left( \sup_j \| \tilde{h}^j \|^4 \right) < \infty. \]

Combining (4.42a), (4.42b), (4.49) and (4.50), we infer by applying the Vitali convergence theorem that
\[ \tilde{v}^j \rightharpoonup \tilde{v} \text{ in } L^2(\tilde{\Omega}; L^2(0, T; V_1)), \]

and
\[ \tilde{h}^j \rightharpoonup \tilde{h} \text{ in } L^2(\tilde{\Omega}; L^2(0, T; V_2)). \]
By thinning the sequence \( j \), if necessary, we conclude that
\[
\| \tilde{\mathbf{v}}^j - \tilde{\mathbf{v}} \| \to 0 \quad \text{and} \quad \| \tilde{h}^j - \tilde{h} \| \to 0,
\]
(4.53)
for almost every \((t, \omega) \in [0, T] \times \tilde{\Omega}\).

For \( \mathbf{v} \in D(-\Delta) \), using (4.53), we readily attain the convergence for the linear terms:
\[
\begin{align*}
\left| \int_0^t \nu \langle \Delta(\tilde{\mathbf{v}}^j - \tilde{\mathbf{v}}), \mathbf{v} \rangle ds \right| & \leq C \| \mathbf{v} \| \left( \int_0^T \| \tilde{\mathbf{v}}^j - \tilde{\mathbf{v}} \|^2 ds \right)^{\frac{1}{2}} \to 0, \\
\left| \int_0^t \delta \langle \Delta(\tilde{h}^j - \tilde{h}), \mathbf{v} \rangle ds \right| & \leq C \| \mathbf{v} \| \left( \int_0^T \| \tilde{h}^j - \tilde{h} \|^2 ds \right)^{\frac{1}{2}} \to 0, \\
\left| \int_0^t \langle g \nabla (\tilde{h}^j - \tilde{h}), \mathbf{v} \rangle ds \right| & \leq C \| \mathbf{v} \| \left( \int_0^T \| \tilde{h}^j - \tilde{h} \|^2 ds \right)^{\frac{1}{2}} \to 0,
\end{align*}
\]
(4.54a,b,c)
\[
\left| \int_0^t \langle f k \times (\tilde{\mathbf{v}}^j - \tilde{\mathbf{v}}), \mathbf{v} \rangle ds \right| \leq C \| \mathbf{v} \| \left( \int_0^T \| \tilde{\mathbf{v}}^j - \tilde{\mathbf{v}} \|^2 ds \right)^{\frac{1}{2}} \to 0.
\]
(4.54d)

By Lemma 4.1 and the Lebesgue Dominated Convergence Theorem, we infer that
\[
E \int_0^T \left( \int_0^t \nu \langle \Delta(\tilde{\mathbf{v}}^j - \tilde{\mathbf{v}}), \mathbf{v} \rangle ds \right) \to 0 \quad \text{as} \quad n \to \infty.
\]
(4.55)

Now for the nonlinear terms, we denote \( \theta(\| \mathbf{v} \|^2 + \| h \|^2) \) by \( \theta(\mathbf{v}, h) \) for the simplicity. As before, we have:
\[
\begin{align*}
\left| \int_0^t \langle P_n[\theta(\tilde{\mathbf{v}}^j, \tilde{h}^j)(\tilde{\mathbf{v}}^j \cdot \nabla)\tilde{\mathbf{v}}^j] - \theta(\tilde{\mathbf{v}}, \tilde{h})(\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}, \mathbf{v} \rangle ds \right|
\leq & \left| \int_0^t \langle \theta(\tilde{\mathbf{v}}^j, \tilde{h}^j)(\tilde{\mathbf{v}}^j \cdot \nabla)\tilde{\mathbf{v}}^j - \theta(\tilde{\mathbf{v}}, \tilde{h})(\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}, \mathbf{v} \rangle ds \right|
\quad + \left| \int_0^t \langle \theta(\tilde{\mathbf{v}}^j, \tilde{h}^j) - \theta(\tilde{\mathbf{v}}, \tilde{h}), (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}, \mathbf{v} \rangle ds \right|
=: J_1^2 + J_2^2.
\end{align*}
\]
We see that:
\[
J_1^2 = \int_0^t \left| \langle \theta(\tilde{\mathbf{v}}^j, \tilde{h}^j)(\tilde{\mathbf{v}}^j \cdot \nabla)\tilde{\mathbf{v}}^j - \theta(\tilde{\mathbf{v}}, \tilde{h})(\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}, \mathbf{v} \rangle ds \right|
\leq C \left( \int_0^T \| \tilde{\mathbf{v}}^j \|^2 + \| \mathbf{v} \|^2 \right) \left( \int_0^T \| \tilde{\mathbf{v}}^j - \tilde{\mathbf{v}} \|^2 dt \right)^{\frac{1}{2}} \to 0,
\]
as \( j \) goes to infinity.
We turn to show that \( J^j_2 \to 0 \) as \( j \to \infty \). For almost every \((t, \omega) \in [0, T] \times \tilde{\Omega}\), we infer from (4.53) that
\[
\|\tilde{\varphi}^j\|^2 + \|\tilde{h}^j\|^2 \to \|\tilde{\varphi}\|^2 + \|\tilde{h}\|^2.
\]
Therefore, we see that \( \theta(\tilde{\varphi}^j, \tilde{h}^j) \to \theta(\tilde{\varphi}, \tilde{h}) \) almost surely. Furthermore, by Hölder’s inequality and Sobolev embedding theorem, we estimate
\[
|\langle \theta(\tilde{\varphi}^j, \tilde{h}^j) - \theta(\tilde{\varphi}, \tilde{h}) | (\tilde{\varphi} \cdot \nabla)\tilde{\varphi}, \tilde{\varphi} \rangle| \leq C\|\tilde{\varphi}\|^2 |\tilde{\varphi}|_{D(-\Delta)}
\]
By Lemma 4.2, we refer that \( \langle \theta(\tilde{\varphi}^j, \tilde{h}^j) - \theta(\tilde{\varphi}, \tilde{h}) | (\nabla)\tilde{\varphi}, \tilde{\varphi} \rangle \in L^2(\Omega \times (0, T)) \). By the Lebesgue Dominated Convergence Theorem, we see that
\[
\mathbb{E} \int_0^T J^j_2 \to 0.
\]
In the same manner, we can show that
\[
\mathbb{E} \int_0^T \int_0^t \langle P_n[\theta(\tilde{\varphi}^j, \tilde{h}^j) \nabla \cdot (\tilde{h}^j \tilde{\varphi}^j)] - \theta(\tilde{\varphi}, \tilde{h}) \nabla \cdot (\tilde{h} \tilde{\varphi}), \tilde{\varphi} \rangle ds \to 0.
\]

The stochastic terms are treated using Lemma 6.4 in the Appendix 6. We apply this lemma with \( S_m = S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W_1, W_2) \). We define
\[
G^m_1, G_1 \in L^2(0, T; L_2(\Omega, H_1)) \text{ and } G^m_2, G_2 \in L^2(0, T; L_2(\Omega, H_2)),
\]
by setting
\[
G^m_1(t) \cdot e_\ell = P_m(\sigma_1(\varphi(t), h(t)) \cdot e_\ell) = \sum_{i=1}^m \sigma^{i\ell}_1(\varphi(t), h(t)) \phi_i,
G^m_2(t) \cdot e_\ell = P_m(\sigma_2(\varphi(t), h(t)) \cdot e_\ell) = \sum_{i=1}^m \sigma^{i\ell}_2(\varphi(t), h(t)) \phi_i,
G_1(t) \cdot e_\ell = \sigma_1(\varphi(t), h(t)) \cdot e_\ell, \quad G_2(t) \cdot e_\ell = \sigma_2(\varphi(t), h(t)) \cdot e_\ell,
\]
for \( \ell = 1, 2, \ldots \)

With (4.42c) at hand, to apply this lemma, we need to show that for \( k = 1, 2 \)
\[
G^m_k(t) \to G_k(t) \text{ in } L^2(0, T; L_2(\Omega, H_k)) \text{ a.s. and thus in probability.}
\]
We have
\[
\int_0^T \|G^m_k(t) - G_m(t)\|_{L_2(\Omega, H_k)}^2 dt = \int_0^T \sum_{\ell=1}^{\infty} |G^m_k(t) \cdot e_\ell - G_k(t) \cdot e_\ell|_{H_k}^2 dt
\]
\[
\leq \int_0^T \sum_{\ell=1}^{\infty} \sum_{i=1}^m |\sigma^{i\ell}_k(\varphi(t), h(t)) \phi_i| - \sum_{i=1}^m |\sigma^{i\ell}_k(\varphi(t), h(t)) \phi_i|_{H_k} \int_0^T \sum_{\ell=1}^{\infty} \sum_{i=m+1}^{\infty} |\sigma^{i\ell}_k(\varphi(t), h(t)) \phi_i|_{H_k}^2 dt
\]
\[
\leq 4 \int_0^T \sum_{\ell=1}^{\infty} \sum_{i=1}^m |\sigma^{i\ell}_k(\varphi(t), h(t)) \phi_i|_{H_k}^2 dt.
\]
By Lipschitz condition (2.9a), we also have

$$
\lim_{T \to \infty} \int_0^T \sum_{i=1}^\infty \sum_{t=1}^\infty \left| \sigma_k^i (\mathbf{v}(t), h(t)) \right|^2_{H_k} \, dt = \int_0^T \sum_{t=1}^\infty |\sigma_k(\mathbf{v}(t), h(t)) - e_k|^2_{H_k} \, dt = \int_0^T \|\sigma_k(\mathbf{v}(t), h(t))\|^2_{L^2(\mathcal{U}, H_k)} \, dt
$$

$$
\leq C \int_0^T (1 + \|\mathbf{v}(t)\|^2 + \|h(t)\|^2) \, dt < \infty.
$$

Which implies that \(\|G_k^m(t)\|^2_{L^2(\mathcal{U}, H_k)}\), \(\|G_k(t)\|^2_{L^2(\mathcal{U}, H_k)}\) are bounded by a function in \(L^1(0, T)\). Since \(G_k^m(t) \cdot e_k = P_n(\sigma_k(\mathbf{v}(t), h(t)) \cdot e_k) \to \sigma_k(\mathbf{v}(t), h(t)) \cdot e_k = G_k(t) \cdot e_k\) in \(H_k\) for a.e. and a.s.

By the Lebesgue dominated convergence theorem we, conclude that

$$
\int_0^T \|G_k^m(t) - G_m(t)\|^2_{L^2(\mathcal{U}, H_k)} \, dt \to 0 \text{ as } m \to \infty.
$$

This shows that \(G_k^m \to G_k\) in probability in \(L^2(0, T; L^2(\mathcal{U}, H_k))\). This convergence along with (4.42c) imply

$$
\int_0^t G_k^m dW_k^m \to G_k dW_k \text{ in probability in } L^2(0, T; X = H_k),
$$

for \(k = 1, 2\).

\(U_p\) to a subsequence, we can have (4.57) holds further for almost sure convergence and hence in \(L^2(\tilde{\Omega}, L^2(0, T; H_k))\) using the Lebesgue dominated convergence theorem.

With all the convergences of the linear and nonlinear and stochastic terms, for every \(\mathbf{v} \in V_1, h \in V_2\) and any measurable set \(K \subset \tilde{\Omega} \times [0, T]\), we have

$$
\mathbb{E} \int_0^T \chi_K \langle \tilde{\mathbf{v}}(t), \mathbf{v} \rangle \, dt = \lim_{k \to \infty} \mathbb{E} \int_0^T \chi_K \langle \tilde{\mathbf{v}}^j, \mathbf{v} \rangle \, dt
$$

$$
= \lim_{k \to \infty} \mathbb{E} \int_0^T \chi_K \langle \tilde{\mathbf{v}}^j, \mathbf{v} \rangle \, dt + \mathbb{E} \int_0^T \chi_K \left( \int_0^t \nu(\Delta \tilde{\mathbf{v}}^j, \mathbf{v}) \, ds \right) \, dt
$$

$$
- \mathbb{E} \int_0^T \chi_K \left( \int_0^t \nu(P_n[\theta(\tilde{\mathbf{v}}, \tilde{\mathbf{h}}^j)(\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}^j + g \nabla \tilde{\mathbf{h}}^j + f \mathbf{k} \times \tilde{\mathbf{v}}^j - F_j], \mathbf{v}) \, ds \right) \, dt
$$

$$
+ \mathbb{E} \int_0^T \chi_K \left( \int_0^t \sum_{k=1}^\infty P_n \sigma_{k,j} \nu(\tilde{\mathbf{h}}^j, \mathbf{v}) d\tilde{\mathbf{w}}_1^{(j,k)} \right) \, dt
$$

$$
= \mathbb{E} \int_0^T \chi_K \langle \tilde{\mathbf{v}}^0, \mathbf{v} \rangle \, dt + \mathbb{E} \int_0^T \chi_K \left( \int_0^t \nu(\tilde{\mathbf{v}}, \mathbf{v}) \, ds \right) \, dt
$$
\[-\mathbb{E} \int_0^T \chi_K \left( \int_0^t \nu(\theta(\tilde{\nu}, \tilde{h})(\tilde{\nu} \cdot \nabla)\tilde{\nu} + g\nabla \tilde{h} + f k \times \tilde{\nu} - F, \nu) \right) dt \]
\[
+ \mathbb{E} \int_0^T \chi_K \left( \int_0^t \sum_{k=1}^\infty (\sigma_k^2(\tilde{\nu}, \tilde{h}), \nu) d\tilde{W}_1^k \right) dt,
\]

and \( \mathbb{E} \int_0^T \chi_K(\tilde{h}(t), \eta) dt = \lim_{k \to \infty} \mathbb{E} \int_0^T \chi_K(\tilde{h}^j, \eta) dt \)
\[
= \lim_{k \to \infty} \left[ \mathbb{E} \int_0^T \chi_K(\tilde{h}^j_0, \eta) dt + \mathbb{E} \int_0^T \chi_K \left( \int_0^t \delta(\Delta \tilde{h}^j, \eta) dt \right) \right] \]
\[
- \mathbb{E} \int_0^T \chi_K \left( \int_0^t \nu(\theta(\tilde{\nu}^j, \tilde{h}^j)\nabla(\tilde{h}^j\tilde{\nu}^j)), \eta) ds \right) dt \]
\[
+ \mathbb{E} \int_0^T \chi_K \left( \int_0^t \sum_{k=1}^\infty P_n \sigma_{2j}^k(\tilde{\nu}^j, \tilde{h}^j), \eta) d\tilde{W}_2^{j,k} \right) dt \]
\[
= \mathbb{E} \int_0^T \chi_K(\tilde{h}_0, \eta) dt + \mathbb{E} \int_0^T \chi_K \left( \int_0^t \nu(\tilde{h}, \eta) ds \right) dt \]
\[
- \mathbb{E} \int_0^T \chi_K \left( \int_0^t \nu(\theta(\tilde{\nu}, \tilde{h})\nabla(\tilde{h}\tilde{\nu})), \eta) ds \right) dt + \mathbb{E} \int_0^T \chi_K \left( \int_0^t \sum_{k=1}^\infty \nu(\sigma_k^2(\tilde{\nu}, \tilde{h}), \nu) d\tilde{W}_2^k \right) dt.\]

Since \( K \) is arbitrary, we see that for a.e. \((\omega, t) \in \bar{\Omega} \times [0, T] \) and every \( \nu \in V_1, \eta \in V_2 \):
\[
\langle \tilde{\nu}(t), \nu \rangle + \int_0^t \langle \nu \Delta \tilde{\nu} + \theta(\nu, \tilde{h})(\nu \cdot \nabla)\nu + g\nabla \tilde{h} + f k \times \nu, \nu \rangle ds = \langle \tilde{\nu}_0, \nu \rangle + \int_0^t \langle F, \nu \rangle ds + \int_0^t \sum_{k=1}^\infty \langle \sigma_k^2(\tilde{\nu}, \tilde{h}), \nu \rangle d\tilde{W}_2^k, \tag{4.58}\]

and
\[
\langle \tilde{h}(t), \eta \rangle + \int_0^t \langle -\delta \Delta \tilde{h} + \theta(\nu, \tilde{h})\nabla(\tilde{h}\tilde{\nu}), \eta \rangle ds = \langle \tilde{h}_0, \eta \rangle + \int_0^t \sum_{k=1}^\infty \langle \sigma_k^2(\tilde{\nu}, \tilde{h}), \nu \rangle d\tilde{W}_2^k. \tag{4.59}\]

Since (4.58) and (4.59) hold for all such \((\nu, \eta) \in V_1 \times V_2 \) so by the density argument, it also holds true for \((\nu, \eta) \in H_1 \times H_2 \). We then obtain (2.23) and (2.24).

The proof of Theorem 4.3 will be complete once we can upgrade the regularity in time for \((\nu, h)\); more precisely, we need to establish \((\nu, h) \in L^2(\Omega, \mathcal{C}([0, T]; V)) \) a.s.

We first introduce some notations to simplify some expositions
\[
\langle U, \tilde{U} \rangle = \langle \nu, \tilde{\nu} \rangle + \langle h, \tilde{h} \rangle, \]
\[
a_0(U, \tilde{U}) = \langle A_0 U, \tilde{U} \rangle = \nu(\nabla \nu, \nabla \tilde{\nu}) + \delta(\nabla h, \nabla \tilde{h}),
\]
where

\[ b_1(v, \bar{v}, \bar{v}) = \int_M (v \cdot \nabla \bar{v}) d\mathcal{M}, \quad b_2(v, \bar{v}, \bar{v}) = \int_M (h \cdot \nabla \bar{v} + \nabla h \cdot \bar{v}) d\mathcal{M}. \]

\[ \sigma(U, \bar{U}) = \langle \sigma_1(U) dW_1, \bar{U} \rangle + \langle \sigma_2(U) dW_2, \bar{U} \rangle = \left( \sum_{k=1}^{\infty} \sigma_1(U) e_k dW_1, \bar{U} \right) + \left( \sum_{k=1}^{\infty} \sigma_2(U) e_k dW_2, \bar{U} \right). \]

Then (4.1a) and (4.1b) can be rewritten in the form

\[ d\langle U, \bar{U} \rangle + a_0(U, \bar{U}) dt + a_1(U, \bar{U}) dt + b(U, U, \bar{U}) dt = \sigma(U, \bar{U}) dW + \langle F, \bar{U} \rangle dt. \] (4.60)

(4.1a) and (4.1b) can also be rewritten in a similar way:

\[ d\langle U, \bar{U} \rangle + a_0(U, \bar{U}) dt + a_1(U, \bar{U}) dt + \theta(\|U\|) b(U, U, \bar{U}) dt = \sigma(U, \bar{U}) dW + \langle F, \bar{U} \rangle dt. \] (4.61)

Now, let \( Z = (Z_1, Z_2) \) be the martingale solution of

\[ d\langle Z_1, Z_2 \rangle + a_0(Z_1, Z_2) dt + a_1(Z_1, Z_2) dt = \sigma(U, \bar{U}) dW + \langle F, \bar{U} \rangle dt, \] (4.62)

where \( U \) is a martingale solution of the modified system (4.1a) and (4.1b). Thanks to Proposition (6.3), (6.4) and theorem (6.10) in [9], we then infer that

\[ Z \in L^2(\bar{\Omega}, L^2(0, T; D(A))) \cap L^2(\bar{\Omega}, C([0, T]; V)). \] (4.63)

Now, take \( \bar{U} = U - Z \), substracting (4.62) from (4.61), we obtain

\[ d\bar{U} + A_1 \bar{U} dt + A_0 \bar{U} dt + B(\bar{U} + Z, \bar{U} + \bar{Z}) dt = 0. \]

Due to (4.45)–(4.48) and (4.63), we see that

\[ \bar{U} \in L^2(\bar{\Omega}, L^2(0, T; D(A))) \cap L^2(\bar{\Omega}, L^\infty(0, T; V)). \] (4.64)

Hence

\[ d\bar{U} = -A_1 \bar{U} - A_0 \bar{U} - \theta(\|\bar{U} + Z\|) B(\bar{U} + Z, \bar{U} + Z) dt \in L^2(\bar{\Omega}, L^2(0, T; H)). \] (4.65)

Form (4.64) and (4.65) and by the classical interpolation Sobolev embedding theorem, we conclude that

\[ \bar{U} \in L^2(\bar{\Omega}, C([0, T]; [D(A), H^{1/2}])) = L^2(\bar{\Omega}, C([0, T]; V)) \text{ a.s.} \] (4.66)

From (4.63) and (4.66), we arrive at

\[ U = \bar{U} + Z \in L^2(\bar{\Omega}, C([0, T]; V)) \cap L^2(\bar{\Omega}, L^2([0, T]; D(A))). \] (4.67)
4.5. Global pathwise uniqueness

Now we prove that the global martingale solution for the modified system is pathwise unique.

**Proposition 4.1.** Suppose that $(\mathcal{S}, \mathbf{v}_1, h_1)$ and $(\mathcal{S}, \mathbf{v}_2, h_2)$ are two global Martingale solutions of (4.1) relative to the same stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W_1, W_2)$. Pathwise uniqueness means that if we define $\Omega_0 := \{ \mathbf{v}_1(0) = \mathbf{v}_2(0), h_1(0) = h_2(0) \}$, then $(\mathbf{v}_1, h_1)$ and $(\mathbf{v}_2, h_2)$ are indistinguishable on $\Omega_0$ in the sense that

$$\mathbb{P}(1_{\Omega_0}(\mathbf{v}_1(t) - \mathbf{v}_2(t)) = 0, \forall t \geq 0) = 1,$$

and

$$\mathbb{P}(1_{\Omega_0}(h_1(t) - h_2(t)) = 0, \forall t \geq 0) = 1.$$

**Proof.** We will let $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, $h = h_1 - h_2$, $\mathbf{v} = 1_{\Omega_0}\mathbf{v}$ and $\tilde{h} = 1_{\Omega_0}h$.

We will also need the following stopping times

$$\tau^{(n)} := \inf_{t \geq 0} \left\{ \int_0^t |\Delta \mathbf{v}_1|^2 ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 |\Delta \mathbf{v}_2|^2 + |\Delta h_1|^2 ||h_1||^2 + |\Delta h_2|^2 ||h_2||^2 ds \geq n \right\}. \quad (4.68)$$

Substituting $\mathbf{v}_1$ and $\mathbf{v}_2$ into (4.1) and taking the difference between these equations, we arrive at the following equations:

$${d}\mathbf{v} - \nu \Delta \mathbf{v} dt + f k \times \mathbf{v} dt + g \nabla h dt + \theta(\mathbf{v}_1, h_1)(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 dt - \theta(\mathbf{v}_2, h_2)(\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2 dt$$

$$= \sum_{k=1}^{\infty} \sigma^k_1(\mathbf{v}_1, h_1)dW^k_1 - \sum_{k=1}^{\infty} \sigma^k_1(\mathbf{v}_2, h_2)dW^k_1 ; \quad (4.69)$$

$$\mathbf{v}(0) = \mathbf{v}_1(0) - \mathbf{v}_2(0)$$

and

$${dh} - \delta \Delta h dt = \theta(\mathbf{v}_2, h_2)\nabla \cdot (h_2 \mathbf{v}_2) dt - \theta(\mathbf{v}_1, h_1)\nabla \cdot (h_1 \mathbf{v}_1) dt + \sum_{k=1}^{\infty} \sigma^k_2(\mathbf{v}_1, h_1)dW^k_2 - \sum_{k=1}^{\infty} \sigma^k_2(\mathbf{v}_2, h_2)dW^k_2; \quad (4.70)$$

$$h(0) = h_1(0) - h_2(0)$$

Applying the Itô formulas in (2.18) and (2.19) with $p = 2$ to both equations (4.69) and (4.70) and adding the corresponding relations together yields the following evolution equations for $||\mathbf{v}||^2$ and $||h||^2$

$${d}||\mathbf{v}||^2 + 2\nu ||\Delta \mathbf{v}||^2 dt + d||h||^2 + 2\delta ||\Delta h||^2 dt$$

$$= -2g( f k \times \mathbf{v}, \Delta \mathbf{v}) dt - 2g(\nabla h, \Delta \mathbf{v}) dt - 2(\theta(\mathbf{v}_2, h_2)(\mathbf{v}_2 \cdot \nabla)\mathbf{v}_2 - \theta(\mathbf{v}_1, h_1)(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1, \Delta \mathbf{v}) dt$$

$$+ \sum_{k=1}^{\infty} (||\sigma^k_1(\mathbf{v}_1, h_1) - \sigma^k_1(\mathbf{v}_2, h_2)||^2 dt + \sum_{k=1}^{\infty} (||\sigma^k_2(\mathbf{v}_1, h_1) - \sigma^k_2(\mathbf{v}_2, h_2)||^2 dt$$

$$+ 2 \sum_{k=1}^{\infty} (\sigma^k_1(\mathbf{v}_1, h_1) - \sigma^k_1(\mathbf{v}_2, h_2), \Delta \mathbf{v})dW^k_1 + 2 \sum_{k=1}^{\infty} (\sigma^k_2(\mathbf{v}_1, h_1) - \sigma^k_2(\mathbf{v}_2, h_2), \Delta h)dW^k_2$$

$$- 2(\theta(\mathbf{v}_2, h_2)\nabla \cdot (h_2 \mathbf{v}_2) - \theta(\mathbf{v}_1, h_1)\nabla \cdot (h_1 \mathbf{v}_1), \Delta h) dt. \quad (4.71)$$
Now we fix \( n \) and stopping times \( \tau_a, \tau_b \) such that \( 0 \leq \tau_a \leq \tau_b \leq \tau^{(n)} \), and we integrate (4.71) in time, multiply by \( 1_{\Omega_0} \), and take the expected value to obtain
\[
E \left( \sup_{t \in [\tau_a, \tau_b]} \| \vec{v} \|^2 + 2\nu \int_{\tau_a}^{\tau_b} |\Delta \vec{v}|^2 \, dt + \sup_{t \in [\tau_a, \tau_b]} \| \vec{h} \|^2 + 2\delta \int_{\tau_a}^{\tau_b} |\Delta \vec{h}|^2 \, dt \right)
\leq E \left( \| \vec{v}(\tau_a) \|^2 + \| \vec{h}(\tau_a) \|^2 \right) + \sum_{i=1}^{10} J_i,
\]
where the \( J_i \) terms are defined and estimated as follows. First, by the Cauchy–Schwarz and Young inequalities, we find
\[
J_1 := 2\nu E \left( \int_{\tau_a}^{\tau_b} |\langle \nabla \vec{h}, \Delta \vec{v} \rangle| \, dt \right) \leq CE \left( \int_{\tau_a}^{\tau_b} \| \vec{h} \|^2 \, dt \right) + \frac{\nu}{2} E \left( \int_{\tau_a}^{\tau_b} |\Delta \vec{v}|^2 \, dt \right).
\]
Next,
\[
J_2 + J_3 := E \left( \int_{\tau_a}^{\tau_b} |\langle \theta(v_1, h_1)(v_1 \cdot v_1)v_1 - \theta(v_2, h_2)(v_2 \cdot v_2)v_2, \Delta \vec{v} \rangle| \, dt \right)
\leq E \int_{\tau_a}^{\tau_b} |\langle (\theta(v_1, h_1) - \theta(v_2, h_2))(v_1 \cdot \nabla)v_1, \Delta \vec{v} \rangle| \, dt + \nu E \int_{\tau_a}^{\tau_b} |\langle (v_1 \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2, \Delta \vec{v} \rangle| \, dt
= J_{23}^1 + J_{23}^2.
\]

\( J_{23}^1 \) and \( J_{23}^2 \) are majored by Hölder’s inequality, Agmon’s inequality, classical Sobolev interpolation:
\[
J_{23}^1 \leq CE \left( \int_{\tau_a}^{\tau_b} (\| \vec{v} \| + \| \vec{h} \|) \| v_1 \|_{L^\infty(M)} \| \nabla v_1 \|_{L^2(M)} \| \Delta \vec{v} \|_{L^2(M)} \frac{3}{2} dt \right)
\leq CE \left( \int_{\tau_a}^{\tau_b} (\| \vec{v} \| + \| \vec{h} \|) \| \Delta v_1 \|^\frac{1}{2}_{L^2(M)} \| \nabla v_1 \|^\frac{3}{2}_{L^2(M)} \| \Delta \vec{v} \|^\frac{3}{2}_{L^2(M)} dt \right)
\leq \nu E \left( \int_{\tau_a}^{\tau_b} |\Delta \vec{v}|^2 \, dt \right) + CE \left( \int_{\tau_a}^{\tau_b} (\| \vec{v} \|^2 + \| \vec{h} \|^2) \| \Delta v_1 \|^2_{L^2(M)} \| v_1 \|^2_{H^1(M)} \, dt \right).
\]

Next we give the estimate for \( J_{23}^2 \):
\[
J_{23}^2 \leq CE \left( \int_{\tau_a}^{\tau_b} |\langle (v_1 \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2, \Delta \vec{v} \rangle| \, dt \right)
\leq CE \left( \int_{\tau_a}^{\tau_b} |\langle (v_1 - v_2) \cdot \nabla)v_1 - (v_2 \cdot \nabla)(v_2 - v_1), \Delta \vec{v} \rangle| \, dt \right)
= CE \left( \int_{\tau_a}^{\tau_b} |\langle \vec{v} \cdot \nabla)v_1 - (v_2 \cdot \nabla)v_2, \Delta \vec{v} \rangle| \, dt \right).
\]
\[
\leq C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} (|\nabla v_1|_{L^2} + |v_2|_{L^2}) \Delta \bar{v} \big|_{L^2} dt \right)
\]

\[
\leq C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} (|\nabla \bar{v}|_{L^2}^2 + |v_1|_{L^2}^2 + |v_2|_{L^2}^2) \big| \Delta v_1 \big|_{L^2} dt \right)
\]

\[
\leq \frac{\nu}{5} C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |\Delta \bar{v}|_{L^2} dt \right) + C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} \|v\|_{L^2}^2 \big| \Delta v_1 \big|_{L^2} + \|v_2\|_{L^2}^2 \big| \Delta v_2 \big|_{L^2} dt \right) \tag{4.75}
\]

The next two deterministic terms are handled in a similar way as follows:

\[
J_4 + J_5 := 2\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |\theta(v_1, h_1) \nabla (h_1 \bar{v}_1) - \theta(v_2, h_2) \nabla (h_2 \bar{v}_2), \Delta \bar{h} \big| dt \right)
\]

\[
\leq 2\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |\theta(v_1, h_1) - \theta(v_2, h_2) \nabla (h_1 \bar{v}_1), \Delta \bar{h} \big| dt \right)
\]

\[
+ 2\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |\theta(v_2, h_2)(\nabla (h_2 \bar{v}_2) - \nabla (h_1 \bar{v}_1)), \Delta \bar{h} \big| dt \right)
\]

\[
\leq \delta\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |\Delta \bar{h}|_{L^2}^2 dt \right) + \frac{\nu}{5} C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |\Delta \bar{v}|_{L^2}^2 dt \right)
\]

\[
+ C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} (\|v\|_{L^2}^2 + \|\bar{h}\|_{L^2}^2) \big( \|v_1\|_{L^2}^2 + \|v_2\|_{L^2}^2 + \|\Delta v_1\|_{L^2}^2 + \|\Delta v_2\|_{L^2}^2 + \|h_1\|_{L^2}^2 + \|h_2\|_{L^2}^2 \big) dt \right) \tag{4.76}
\]

Clearly, by the Cauchy Schwarz inequality,

\[
J_6 := 2\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |(f \mathbf{k} \times \bar{v}, \Delta \bar{v})| dt \right) \leq \frac{\nu}{5} C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} |\Delta \bar{v}|_{L^2}^2 dt \right) + C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} \|v\|_{L^2}^2 dt \right). \tag{4.77}
\]

For the next two terms, we simply use the Lipschitz assumption (2.9)

\[
J_7 + J_8 := \sum_{i=1}^{2} \mathbb{E}\left( \int_{\tau_a}^{\tau_b} \sum_{k=1}^{\infty} \sigma_i(v_1, h_1) \mathbf{e}_k - \sigma_i(v_2, h_2) \mathbf{e}_k \big| dt \right) \leq C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} (\|v\|_{L^2}^2 + \|\bar{h}\|_{L^2}^2) \big) dt \right. \tag{4.78}
\]

The estimates for the last two stochastic terms are obtained by using the BDG inequality and the results are similar to (3.12) and (3.13)

\[
J_9 := 2\mathbb{E}\left( \sup_{t \in [\tau_a, \tau_b]} \left| \int_{\tau_a}^{\tau_b} \sum_{k=1}^{\infty} \sigma_1^k(v_1, h_1) - \sigma_1^k(v_2, h_2), \Delta \bar{v}) dW_1^k \right| \right)
\]

\[
\leq \frac{1}{2} \mathbb{E}\left( \sup_{t \in [\tau_a, \tau_b]} \|v\|_{L^2}^2 \right) + C\mathbb{E}\left( \int_{\tau_a}^{\tau_b} (\|v\|_{L^2}^2 + \|\bar{h}\|_{L^2}^2) dt \right), \tag{4.79}
\]
\[ J_{10} := 2E \left( \sup_{t \in [\tau_n, \tau_h]} \left| \int_{\tau_n}^{\infty} \sum_{k=1}^{\infty} (\sigma^k_2(v_1, h_1) - \sigma^k_2(v_2, h_2), \Delta \tilde{h}) dW_k^k \right| \right) \]

\[ \leq \frac{1}{2} E \left( \sup_{t \in [\tau_n, \tau_h]} \| \tilde{h} \|^2 \right) + C E \left( \int_{\tau_n}^{\tau_h} (\| \tilde{v} \|^2 + \| \tilde{h} \|^2) dt \right). \]  

(4.80)

Rearranging (4.72)–(4.80) and multiplying by two, we obtain

\[ E \left( \sup_{t \in [\tau_n, \tau_h]} \| \tilde{v} \|^2 + \nu \int_{\tau_n}^{\tau_h} | \Delta \tilde{v} |^2 dt + \sup_{t \in [\tau_n, \tau_h]} \| \tilde{h} \|^2 + \delta \int_{\tau_n}^{\tau_h} | \Delta \tilde{h} |^2 dt \right) \]

\[ \leq 2E \left( (\| \tilde{v}(\tau_n) \|^2 + \| \tilde{h}(\tau_n) \|^2) \right) \]

\[ + \frac{C}{2} E \left( (\| \tilde{v} \|^2 + \| \tilde{h} \|^2)(1 + | \Delta v_1 |^2 \| v_1 \|^2 + | \Delta v_2 |^2 \| v_2 \|^2 + | h_1 |^2 \| h_1 \|^2 + | h_2 |^2 \| h_2 \|^2) \right) dt. \]

(4.81)

Now, we apply the stochastic Gronwall inequality (Lemma 6.3 with \( X = \| \tilde{v} \|^2 + \| \tilde{h} \|^2, Y = \nu \| \tilde{v} \|^2 + \delta \| \tilde{h} \|^2, Z = 0, R = 1 + | \Delta v_1 |^2 \| v_1 \|^2 + | \Delta v_2 |^2 \| v_2 \|^2 + | h_1 |^2 \| h_1 \|^2 + | h_2 |^2 \| h_2 \|^2 \)) to obtain

\[ E \left( \sup_{t \in [0, \tau^{(n)}]} \| \tilde{v} \|^2 + \nu \int_{0}^{\tau^{(n)}} | \Delta \tilde{v} |^2 dt + \sup_{t \in [0, \tau^{(n)}]} \| \tilde{h} \|^2 + \delta \int_{0}^{\tau^{(n)}} | \Delta \tilde{h} |^2 dt \right) \]

\[ \leq C E \left( (\| \tilde{v}(0) \|^2 + \| \tilde{h}(0) \|^2) \right) = 0. \]

(4.82)

From the definition of \( \tau^{(n)} \) it is easy to see that \( \tau^{(n)} \) is an increasing sequence and by Lemma 4.1, we see that \( \lim_{n \to \infty} \tau^{(n)} = \infty \). Thus we have shown,

\[ E \left( \sup_{t \in [0, T]} (\| \tilde{v} \|^2 + \| \tilde{h} \|^2) \right) = 0. \]

(4.83)

This implies that

\[ \mathbb{P}(\mathbb{1}_{\Omega_0}(v_1(t) - v_2(t)) = 0; \forall t \geq 0) = 1, \]

(4.84)

and \( \mathbb{P}(\mathbb{1}_{\Omega_0}(h_1(t) - h_2(t)) = 0; \forall t \geq 0) = 1. \)

(4.85)

In other words, \( v_1 \) and \( v_2 \) are indistinguishable on \( \Omega_0 \) and so are \( h_1 \) and \( h_2 \). This proves global pathwise uniqueness. \( \square \)

4.6. Compactness revisited

Having established the existence of martingale solutions and pathwise uniqueness for the modified system (4.1), we may apply the Gyöngy–Krylov theorem (see [20]), which is the infinite dimensional extension of the Yamada–Watanabe Theorem (see [34]), to infer the existence of a global pathwise solution \((v, h)\). To do so, we return to the sequence \( \{(v^n, h^n)\} \) of Galerkin solutions relative to the given stochastic basis \( \mathcal{S} \). We argue in a similar manner to [10] by considering the collections of joint distributions \( \mu^{m,n}_{v} := \mu_{v} \times \mu_{v}^{n} \) and \( \mu^{m,n}_{h} := \mu_{h}^{m} \times \mu_{h}^{n} \). We define the extended phase spaces

\[ \mathcal{X} := \mathcal{X}_\nu \times \mathcal{X}_\nu \times \mathcal{X}_h \times \mathcal{X}_h \times \mathcal{X}_{W_1} \times \mathcal{X}_{W_2}, \quad \mathcal{X}^{(n)} := \mathcal{X}_\nu \times \mathcal{X}_\nu \times \mathcal{X}_h \times \mathcal{X}_h. \]

(4.86)
As above in (4.28) and (4.29) we get

$$
\mu_v^n(E) = Pr(v^n \in E) \text{ for } E \in \mathcal{X}_v,
$$

$$
\mu_h^n(E) = Pr(h^n \in E) \text{ for } E \in \mathcal{X}_h,
$$

$$
\mu_W^n(E) = Pr(W^n \in E) \text{ for } E \in \mathcal{X}_W.
$$

We then take

$$
\nu^{m,n} = \mu_v^m \times \mu_v^n \times \mu_h^m \times \mu_h^n \times \mu_W^m \times \mu_W^n.
$$

**Lemma 4.5.** The collection \( \{ \nu^{m,n} \} \) is tight and hence pre-compact on \( \mathcal{X}^J \).

**Proof.** The proof follows exactly that of Lemma 4.4. We take \( B_R^1, B_R^2, B_R^3 \) and \( B_R^4 \) as in Lemma 4.4. We can therefore choose \( A^v, A^h, \tilde{A}^v, \tilde{A}^h \) compact in \( \mathcal{X}_v, \mathcal{X}_u, \mathcal{X}_W, \mathcal{X}_W, \) respectively, so that

$$
\mu_v^n(A^v) \geq 1 - \frac{\epsilon}{4}, \mu_v^n(A^h) \geq 1 - \frac{\epsilon}{4}, \mu_W^n(A^v) \geq 1 - \frac{\epsilon}{2}, \mu_W^n(A^h) \geq 1 - \frac{\epsilon}{2} \tag{4.87}
$$

We take \( A_\epsilon = A^v \times A^v \times A^h \times A^h \times \tilde{A}^v \times \tilde{A}^h \) which is compact in \( \mathcal{X}^J \), and by (4.87) we see that

$$
\nu^{m,n}(A_\epsilon) \geq \left(1 - \frac{\epsilon}{4}\right)^4 \left(1 - \frac{\epsilon}{2}\right)^2 \geq 1 - \epsilon,
$$

which holds for all \( 0 < \epsilon < 1 \). The proof of the lemma is complete. \( \square \)

We now suppose that \( \{ \nu^{m_k,n_k} \}_{k \geq 0} \) is any subsequence. By the above lemma, this sequence is tight and hence by Proposition 6.1 (Prohorov’s Theorem), we may choose the subsequence \( k' \) so that \( \nu^{m_k,n_k} \) converges weakly to an element \( \nu' \). Then by Proposition 6.2 (Skorohod’s Theorem), we infer the existence of a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) upon which is defined a sequence of random elements \( (\tilde{v}^{m_k}, \tilde{v}^{n_k}, \tilde{h}^{m_k}, \tilde{h}^{n_k}, \tilde{W}_1, \tilde{W}_2) \) converging a.s. in \( \mathcal{X}^J \) to an element \( (\tilde{v}, \tilde{v}, \tilde{h}, \tilde{h}, \tilde{W}_1, \tilde{W}_2) \) in such a way that

$$
\tilde{P} \left( (\tilde{v}^{m_k}, \tilde{v}^{n_k}, \tilde{h}^{m_k}, \tilde{h}^{n_k}, \tilde{W}_1, \tilde{W}_2) \in \cdot \right) = \nu^{m_k,n_k}(\cdot),
$$

$$
\tilde{P} \left( (\tilde{v}, \tilde{v}, \tilde{h}, \tilde{h}, \tilde{W}_1, \tilde{W}_2) \in \cdot \right) = \nu'(\cdot). \tag{4.88}
$$

Note that in particular \( \mu_v^{m_k,n_k} \) and \( \mu_h^{m_k,n_k} \) converge weakly to \( \mu_v \) and \( \mu_h \), respectively, defined by

$$
\mu_v(\cdot) := \tilde{P} \left( (\tilde{v}, \tilde{v}) \in \cdot \right) \quad \text{and} \quad \mu_h(\cdot) := \tilde{P} \left( (\tilde{h}, \tilde{h}) \in \cdot \right). \tag{4.89}
$$

We then infer exactly as in the preceding section that both \( (\tilde{v}, \tilde{h}) \) and \( (\tilde{v}, \tilde{h}) \) are martingale solutions over the same stochastic basis. One can easily prove that these solutions agree with each other at time \( t = 0 \) a.s., and hence, by uniqueness, we see that \( (\tilde{v}, \tilde{h}) = (\tilde{v}, \tilde{h}) \) in \( \mathcal{X}_v \times \mathcal{X}_h \) a.s. In other words,

$$
\nu' \left( \{(x_1, x_2, y_1, y_2) \in \mathcal{X}^J_{v,h} : x_1 = x_2, y_1 = y_2\} \right) = \tilde{P} \left( (\tilde{v}, \tilde{h}) = (\tilde{v}, \tilde{h}) \right) = 1. \tag{4.90}
$$

This implies, by Proposition 6.3, that the original sequence \( (v^n, h^n) \) defined on the initial probability space \((\Omega, \mathcal{F}, P)\) converges to an element \( (v, h) := (\tilde{v}, \tilde{h}) \), in the topology of \( \mathcal{X} \), i.e.
\[ \mathbf{v}^n \to \mathbf{v} \quad \text{a.s. in } L^2(0, T; V_1) \cap C([0, T]; H_1), \]
\[ h^n \to h \quad \text{a.s. in } L^2(0, T; V_2) \cap C([0, T]; H_2). \]

By applying Section 4.5, we may conclude that \((\mathbf{v}, h)\) is a global pathwise solution of (4.1).

5. Existence and uniqueness of solutions for the original system

5.1. Local martingale solutions

Theorem 4.3 already shows that \((\tilde{S}, \tilde{V}, \tilde{h})\) is a global Martingale solution for (4.1). Now we set
\[ \tau := \inf_{t \geq 0} \left\{ \sup_{0 \leq r \leq t} (\| \tilde{V}(r) \|^2 + \| \tilde{h}(r) \|^2) > M \right\} \wedge T \]
where \(M = 1 + \| \tilde{V}_0 \|^2 + \| \tilde{h}_0 \|^2 \).

By the following lemma, \(\tau\) is strictly positive, and we observe that
\[ \int_0^{t \wedge \tau} \theta(\tilde{V}, \tilde{h}) \nabla \cdot (\tilde{h} \tilde{V}) ds = \int_0^{t \wedge \tau} \nabla \cdot (\tilde{h} \tilde{V}) ds + \int_0^{t \wedge \tau} \theta(\tilde{V}, \tilde{h}) (\tilde{V} \cdot \nabla) \tilde{V} ds = \int_0^{t \wedge \tau} (\tilde{V} \cdot \nabla) \tilde{V} ds. \]

We obtain that \((\tilde{S}, \tilde{V}, \tilde{h}, \tilde{\tau})\) is a local martingale solution. The proof of Theorem 2.1 is complete.

Lemma 5.1. The stopping time \(\tau\) defined in (5.1) is strictly positive almost surely.

Proof. To simplifying the writing, we will write \(\mathbf{v}\) and \(h\) instead of \(\tilde{V}\) and \(\tilde{h}\) from now on. We then integrate (4.13) from 0 to \(s\) and take the supremum for \(s\) over \([0, \tau \wedge \epsilon]\) to obtain
\[ \sup_{s \in [0, \tau \wedge \epsilon]} \| \mathbf{v} \|^2 + \nu \int_0^{\tau \wedge \epsilon} \| \mathbf{v} \|^2 dt + \sup_{s \in [0, \tau \wedge \epsilon]} \| h \|^2 + \delta \int_0^{\tau \wedge \epsilon} \| h \|^2 dt \]
\[ \leq \| \mathbf{v}(0) \|^2 + \| h(0) \|^2 + C \int_0^{\tau \wedge \epsilon} (1 + |F|^2 + \| \mathbf{v} \|^2 + \| h \|^2) dt \]
\[ + 2 \sup_{s \in [0, \tau \wedge \epsilon]} \left| \int_0^s \sum_{k=1}^{\infty} \langle \sigma_1^k(\mathbf{v}, h), \Delta \mathbf{v} \rangle dW_1^k \right| + 2 \sup_{s \in [0, \tau \wedge \epsilon]} \left| \int_0^s \sum_{k=1}^{\infty} \langle \sigma_2^k(\mathbf{v}, h), \Delta h \rangle dW_2^k \right|, \]
which yields
\[ \sup_{s \in [0, \tau \wedge \epsilon]} \| \mathbf{v} \|^2 + \sup_{s \in [0, \tau \wedge \epsilon]} \| h \|^2 - \| \mathbf{v}(0) \|^2 - \| h(0) \|^2 \]
\[ \leq C \int_0^{\tau \wedge \epsilon} (1 + |F|^2 + \| \mathbf{v} \|^2 + \| h \|^2) dt \]
\[ + 2 \sup_{s \in [0, \tau \wedge \epsilon]} \left| \int_0^s \sum_{k=1}^{\infty} \langle \sigma_1^k(\mathbf{v}, h), \Delta \mathbf{v} \rangle dW_1^k \right| + 2 \sup_{s \in [0, \tau \wedge \epsilon]} \left| \int_0^s \sum_{k=1}^{\infty} \langle \sigma_2^k(\mathbf{v}, h), \Delta h \rangle dW_2^k \right|. \]
This implies
\[
\mathbb{P}\left( \sup_{s \in [0, \tau \wedge \epsilon]} \|v\|^2 + \sup_{s \in [0, \tau \wedge \epsilon]} \|h\|^2 - \|v(0)\|^2 - \|h(0)\|^2 > 1 \right)
\leq \mathbb{P}\left( C \int_0^{\tau \wedge \epsilon} (1 + |F|^2 + \|v\|^2 + \|h\|^2) dt > \frac{1}{3} \right)
\leq \mathbb{P}\left( 2 \sup_{s \in [0, \tau \wedge \epsilon]} \left| \int_0^s \sum_{k=1}^\infty \langle \sigma_1^k(v, h), \Delta v \rangle dW_1^k \right| > \frac{1}{3} \right) + \mathbb{P}\left( \int_0^\tau \sum_{k=1}^\infty \langle \sigma_2^k(v, h), \Delta h \rangle dW_2^k \right) > \frac{1}{3})
=: P_1 + P_2 + P_3.
\]
(5.4)

By Chebyshev’s inequality, Lemma 4.1 and the additional assumptions that \( F \in L^\infty(0, T; H_1) \), we estimate the first term on the right hand side as follows:
\[
P_1 \leq 3CE\left( \int_0^{\tau \wedge \epsilon} (1 + |F|^2 + \|v\|^2 + \|h\|^2) dt \right)
\leq 3CE\left( \epsilon \sup_{s \in [0, \tau \wedge \epsilon]} (1 + |F(s)|^2 + \|v(s)\|^2 + \|h(s)\|^2) \right)
\leq 3CE \left( 1 + |F|_{L^\infty(0, T; H_1)} + 2K_1 \right)
\leq C \cdot \epsilon,
\]
(5.5)

where \( C \) is a constant depending only on the initial datum and \( |F|_{L^2(\Omega; L^\infty(0, T; H_1))} \).

For the second term, we use Chebyshev’s inequality and (4.15) with \( \tau_a = 0 \) and \( \tau_b = \tau \wedge \epsilon \) to obtain
\[
P_2 \leq 6E\left( \sup_{s \in [0, \tau \wedge \epsilon]} \left| \int_0^s \sum_{k=1}^\infty \langle \sigma_1^k(v, h), \Delta v \rangle dW_1^k \right| \right)
\leq 6E\left( \int_0^{\tau \wedge \epsilon} \|v\|^2(1 + \|v\|^2 + \|h\|^2) dt \right)^{1/2}
\leq 6E\left( \epsilon \sup_{s \in [0, \tau \wedge \epsilon]} \|v(s)\|^2(1 + \|v(s)\|^2 + \|h(s)\|^2) \right)
\leq 6\epsilon(K_1(2K_1 + 1))
\leq C \cdot \epsilon.
\]
(5.6)

For the third term, we use Chebyshev’s inequality and BDG inequality with \( \tau_a = 0 \) and \( \tau_b = \tau \wedge \epsilon \) to find
\[
P_3 \leq 6E\left( \sup_{s \in [0, \tau \wedge \epsilon]} \left| \int_0^s \sum_{k=1}^\infty \langle \sigma_2^k(v, h), \Delta h \rangle dW_2^k \right| \right)
\leq 6E\left( \int_0^{\tau \wedge \epsilon} \|h\|^2(1 + \|v\|^2 + \|h\|^2) dt \right)^{1/2}
\leq 6E\left( \epsilon \sup_{s \in [0, \tau \wedge \epsilon]} \|h(s)\|^2(1 + \|v(s)\|^2 + \|h(s)\|^2) \right)
\]
(5.7)
\[
\leq 6\varepsilon(K_1(2K_1 + 1)) \\
\leq C \cdot \varepsilon.
\]

Combining (5.5), (5.6), and (5.7), we have

\[
P\left( \sup_{s \in [0, T \wedge \varepsilon]} \|v(s)\|^2 + \sup_{s \in [0, T \wedge \varepsilon]} \|h(s)\|^2 - \|v(0)\|^2 - \|h(0)\|^2 > 1 \right) \leq C \cdot \varepsilon. \tag{5.8}
\]

This implies

\[
P(\tau = 0) = P\left( \bigcap_{\varepsilon > 0} \{ \tau < \varepsilon \} \right) = \lim_{\varepsilon \to 0} P\{ \tau < \varepsilon \}
\leq \lim_{\varepsilon \to 0} P\left( \sup_{s \in [0, T \wedge \varepsilon]} \|v(s)\|^2 + \sup_{s \in [0, T \wedge \varepsilon]} \|h\|^2 - \|v_0\|^2 - \|h_0\|^2 > 1 \right)
\leq \lim_{\varepsilon \to 0} C \cdot \varepsilon = 0.
\]

Therefore,

\[
P(\tau > 0) = 1. \quad \square \tag{5.10}
\]

5.2. Local pathwise solutions

We let \( \tau \) be as in (5.1), and use an identical argument to Section 4.6 to conclude that \((v, h, \tau)\) is a local pathwise solution of (1.1).

5.3. Maximal pathwise solutions

We also see that the local solution can be extended in time to be a maximal solution.

**Proposition 5.1.** There exists a unique maximal solution \((v, h, \xi)\) and a sequence \(\rho_R\) announcing \(\xi\).

**Proof.** With the uniqueness already proved, we consider the set \(\mathcal{L}\) of all stopping times such that \(\tau \in \mathcal{L}\) if and only if there exist processes \((v, h)\) s.t. \((v, h, \tau)\) is a local pathwise solution. Clearly if two stopping times are in \(\mathcal{L}\), then so is their maximum and if \(\sigma \in \mathcal{L}\), then so is \(\rho \wedge \sigma\) where \(\rho\) is any stopping time. Let \(\xi = \sup \mathcal{L}\) and choose an increasing sequence \(\tau_k \in \mathcal{L}\) such that \(\tau_k \to \xi\) a.s.

For each \(\tau_k\), denote by \((v_k, h_k)\) the corresponding process that makes \((v_k, h_k, \tau_k)\) a local pathwise solution. Let

\[
\Omega_{k,k'} = \{v_k(t \wedge \tau_k \wedge \tau_{k'}) = v_{k'}(t \wedge \tau_k \wedge \tau_{k'}), h_k(t \wedge \tau_k \wedge \tau_{k'}) = h_{k'}(t \wedge \tau_k \wedge \tau_{k'}); t \geq 0\}. \tag{5.11}
\]

Then, by uniqueness, we see that \(\bar{\Omega} = \cap_{k,k'} \Omega_{k,k'}\) is a set of full measure. For \(\omega\) on this set and every \(t > 0\), the sequence \(\{v_k(t \wedge \tau_k)1_{t < \xi}, h_k(t \wedge \tau_k)1_{t < \xi}\}\) is Cauchy in \(H_1 \times H_2\). Let

\[
\bar{v}(t) = \lim_{k \to \infty} v_k(t \wedge \tau_k)1_{t < \xi} \quad \text{and} \quad \bar{h}(t) = \lim_{k \to \infty} h_k(t \wedge \tau_k)1_{t < \xi} \quad \text{a.s.} \tag{5.12}
\]
Then for any $T > 0$, we have
\[
\mathbb{E}\left( \sup_{t \in [0, \xi \wedge T]} \| \dot{v} \|^2 + \int_0^{\xi \wedge T} |\Delta \dot{v}|^2 dt + \sup_{t \in [0, \xi \wedge T]} \| \dot{h} \|^2 + \int_0^{\xi \wedge T} |\Delta \dot{h}|^2 dt \right) < \infty. \tag{5.13}
\]

We may then define $v(t) \in H_1$, $h(t) \in H_2$ by:
\[
\begin{align*}
(v(t), v) + \int_0^{t \wedge \xi} (-\nu \Delta \dot{v} + (\dot{\nu} \cdot \nabla) \dot{v} + g \nabla \dot{h} + f \kappa \times \dot{v}, v) ds &= (v_0, v) + \int_0^{t \wedge \xi} (F, v) ds + \int_0^{t \wedge \xi} \langle \sigma_1(\dot{v}, \dot{h}), v \rangle dW^k_1, \\
(h(t), \eta) + \int_0^{t \wedge \xi} (-\delta \dot{h} + \nabla \cdot (\dot{h} \dot{v}), \eta) dt &= (h_0, \eta) + \int_0^{t \wedge \xi} \langle \sigma_2(\dot{v}, \dot{h}, \eta) \rangle dW^k_2, \tag{5.15}
\end{align*}
\]
for any $t > 0$, $v \in H_1$, $\eta \in H_2$. Clearly for $t < \xi(\omega)$, $v(t, \omega) = \tilde{v}(t, \omega)$, $h(t, \omega) = \tilde{h}(t, \omega)$, and $v$ and $h$ are weakly continuous a.s. in $H_1$ and $H_2$, respectively. Thus, $(v, h, \xi)$ is a local pathwise solution.

For $R > 0$, define the stopping time
\[
\rho_R := \inf_{t \geq 0} \left\{ \sup_{s \in [0, t]} \| v \|_v^2 + \int_0^t |\Delta v|^2 ds + \sup_{s \in [0, t]} \| h \|_h^2 + \int_0^t |\Delta h|^2 dt > R \right\} \wedge \xi. \tag{5.16}
\]
Then $(v, h, \rho_R)$ is a local pathwise solution for any $R > 0$ and $\{\rho_R\}_{R \geq 0}$ announces $\xi$. \hfill \Box

To conclude, we have completed the proof of Theorem 2.2.

**Remark 5.1.** Using a similar approach, we successfully expand our work to systems of parabolic SPDEs with polynomial growth rates, see our coming work [22].

6. Appendices

6.1. Appendix A

Suppose that $\hat{H}$ is a separable Hilbert space. Given $p \geq 2$, $\alpha \in (0, 1)$, we define the fractional derivative space $W^{\alpha,p}(0, T; \hat{H})$ as the Sobolev space of all $u \in L^p(0, T; \hat{H})$ such that
\[
\int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t-s|^{1+\alpha p}} dtds < \infty, \tag{6.1}
\]
endowed with the norm
\[
|u|^p_{W^{\alpha,p}(0, T; \hat{H})} = \int_0^T |u(t)|^p dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t-s|^{1+\alpha p}} dtds. \tag{6.2}
\]
For the case $\alpha = 1$, we take $W^{1,p}([0,T];X) = \{ u \in L^p([0,T];X) : \frac{du}{dt} \in L^p([0,T];X) \}$ to be the classical Sobolev space with it usual norm.\footnote{\[ \| u \|_{W^{1,p}([0,T];X)} = \int_0^T |u(s)|_X^p \, ds + \int_0^T \| \frac{du}{dt}(s) \|_X^p \, ds. \] Note that for $\alpha \in (0,1)$, $W^{1,p}([0,T];X) \subset W^{\alpha,p}([0,T];X)$ and $\| u \|_{W^{\alpha,p}([0,T];X)} \leq C \| u \|_{W^{1,p}([0,T];X)}$.}

We have applied the following lemmas, the proofs of which can be found in e.g. \cite{13} and \cite{28}:

**Lemma 6.1.** Let $E_0 \subset \subset E \subset E_1$ be Banach spaces with the injections being continuous and $E_0$, $E_1$ reflexive. For $p \in (1, \infty)$, $\alpha \in (0,1)$, we have

\[
L^p (0,T;E_0) \cap W^{\alpha,p} (0,T;E_1) \subset \subset L^p (0,T;E).
\] (6.3)

**Lemma 6.2.** If $E \subset \subset \tilde{E}$ are Banach spaces and $p \in (1, \infty)$, $\alpha \in (0,1]$ are such that $\alpha p > 1$, then

\[
W^{\alpha,p} (0,T;E) \subset \subset C ([0,T];\tilde{E}).
\] (6.4)

We additionally often use the following stochastic version of the Gronwall lemma (see e.g. \cite{19}):

**Lemma 6.3.** Fix $T > 0$ and assume that $X,Y,Z,R : \Omega \times [0,T) \rightarrow \mathbb{R}$ are non-negative stochastic processes. Let $\tau < T$ be a stopping time such that

\[
\mathbb{E}\left( \int_0^{\tau} (RX + Z) ds \right) < \infty \quad \text{and} \quad \int_0^{\tau} R ds < \kappa, \quad a.s.
\]

Suppose that for all stopping times $0 \leq \tau_a \leq \tau_b \leq \tau$

\[
\mathbb{E}\left( \sup_{t \in [\tau_a,\tau_b]} X + \int_{\tau_a}^{\tau_b} Y ds \right) \leq C_0 \mathbb{E}\left( X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds \right),
\]

where $C_0$ is independent of $\tau_a$ and $\tau_b$. Then

\[
\mathbb{E}\left( \sup_{t \in [0,\tau]} X + \int_0^{\tau} Y ds \right) \leq C \mathbb{E}\left( X(0) + \int_0^{\tau} Z ds \right),
\]

where $C$ is a constant depending only on $C_0$, $T$, and $\kappa$.

Finally, we require the Vitali convergence theorem (see e.g. \cite{16}):

**Theorem 6.1.** Suppose that a sequence of functions $\{f_n\}$ are $L^p$ integrable on a finite measure space, where $1 \leq p < \infty$. Then this sequence converges in $L^p$ to a measurable function $f$ if the following conditions are satisfied:

(i) $\{f_n\}$ converges to $f$ in measure; and

(ii) the functions $\{|f_n|^p\}$ are uniformly integrable.

**Remark 6.1.** One can easily prove for $p > 1$ and a nonempty family $\mathcal{X}$ of random variables bounded in $L^p$ that if $\sup_{X \in \mathcal{X}} \|X\|_{L^p} < \infty$, then $\mathcal{X}$ is uniformly integrable.
6.2. Appendix B

**Definition 6.1.** Suppose \((X,d)\) is a complete separable metric space with \(\mathcal{B}(X)\) its associated Borel \(\sigma\)-algebra. Let \(C_b(X)\) be the set of all real-valued continuous bounded functions on \(X\), and let \(\mathcal{P}(X)\) be the set of all probability measures on \((X,\mathcal{B}(X))\). A collection \(\Lambda \subset \mathcal{P}(X)\) is tight if for every \(\epsilon > 0\) there exists a compact set \(K_\epsilon \subset X\) s.t.

\[
\mu(K_\epsilon) \geq 1 - \epsilon \quad \forall \mu \in \Lambda.
\]  

(6.5)

A sequence \(\{\mu_n\}_{n \geq 0} \subset \mathcal{P}(X)\) converges weakly to a probability measure \(\mu\) if

\[
\int fd\mu_n \to \int fd\mu \quad \forall f \in C_b(X).
\]

(6.6)

The proofs of the following results can be found in e.g. [9].

**Proposition 6.1** (Prohorov’s Theorem). A collection \(\Lambda \subset \mathcal{P}(X)\) is weakly compact if and only if it is tight.

**Proposition 6.2** (Skorohod representation theorem). Suppose that a sequence \(\{\mu_n\}_{n \geq 0}\) converges weakly to a measure \(\mu\). Then there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and a sequence of \(X\)-valued random variables \(\{\tilde{Y}_n\}_{n \geq 0}\) relative to this space such that \(\tilde{Y}_n\) converges a.s. to the random variable \(\tilde{Y}\) and such that the laws of \(\tilde{Y}_n\) and \(\tilde{Y}\) are \(\mu_n\) and \(\mu\), respectively, i.e. \(\mu_n(E) = \mathbb{P}(\tilde{Y}_n \in E)\), \(\mu(E) = \mathbb{P}(\tilde{Y} \in E)\), \(\forall E \in \mathcal{B}(X)\).

Finally, we suppose that \(\{Y_n\}_{n \geq 0}\) is a sequence of \(X\)-valued random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \(\{\mu_{m,n}\}_{m,n \geq 0}\) be the collection of joint laws of \(\{Y_n\}_{n \geq 0}\), i.e.

\[
\mu_{m,n}(E) := \mathbb{P}(\{Y_m, Y_n\} \in E), \quad \forall E \in \mathcal{B}(X \times X).
\]  

(6.7)

We also need this result from [20]:

**Proposition 6.3** (Gyöngy-Krylov Theorem). A sequence of \(X\)-valued random variables \(\{Y_n\}_{n \geq 0}\) converges in probability if and only if for every subsequence of joint probability laws, \(\{\mu_{m_k,n_k}\}_{k \geq 0}\) there exists a further subsequence which converges weakly to a probability measure \(\mu\) s.t.

\[
\mu(\{(x, y) \in X \times X : x = y\}) = 1.
\]

(6.8)

**Lemma 6.4.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a fixed probability space, \(X\) a separable Hilbert space. Consider a sequence of stochastic bases \(\mathcal{S}_n = (\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_{t \geq 0}, \mathbb{P}, W_1^n, W_2^n)\), where each \(W_t^n\) is a cylindrical Brownian motion over \(\Omega\) with respect to \(\mathcal{F}_t^n\). Assume that \(\{G^n\}_{n \geq 0}\) are a collection of \(X\)-valued \(\mathcal{F}_t^n\) predictable processes such that \(G^n \in L^2(0, T; L_2(\Omega, X))\) a.s. Finally, consider \(\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W_1, W_2)\) and \(G \in L^2(0, T; L_2(\Omega, X))\) a.s., which is \(\mathcal{F}_t\) predictable. If

\[
G^n \to G \quad \text{in probability in } L^2(0, T; L_2(\Omega, H)),
\]

(6.9)

\[
W^n \to W \quad \text{in probability in } C([0, T]; \mathfrak{H}_0),
\]

(6.10)

then

\[
\int_0^t G^n dW^n \to \int_0^t GdW \quad \text{in probability in } L^2(0, T; X).
\]

(6.11)
6.3. Appendix C

In this appendix, we briefly consider the deterministic problem \((\sigma_i \equiv 0 \text{ for } i = 1, 2)\) in the case of the boundary condition (1.2), (1.3). Our goal is to show why we use these particular boundary conditions and to derive the variational formulation which is used in the Itô formula. We begin with (1.1a) which we multiply by \(\tilde{v} \in V\) and integrate over \(\mathcal{M}\):

\[
\left( \frac{\partial \mathbf{v}}{\partial t}, \tilde{v} \right) - \nu(\Delta \mathbf{v}, \tilde{v}) + ((\mathbf{v} \cdot \nabla) \mathbf{v}, \tilde{v}) + g(\nabla h, \tilde{v}) + (f \mathbf{k} \times \mathbf{v}, \tilde{v}) = (F, \tilde{v}).
\] 

(6.12)

The second term can be calculated as:

\[
-\nu(\Delta \mathbf{v}, \tilde{v}) = \nu(\text{curl(curl}(\mathbf{v})), \tilde{v}) - \nu(\nabla(\nabla \cdot \mathbf{v}), \tilde{v})
\]

\[
= \nu(\text{curl}(\mathbf{v}), \text{curl}(\tilde{v})) - \nu \int_{\partial \mathcal{M}} \text{curl}(\mathbf{v}) \cdot (\tilde{v} \wedge n) dS
\]

\[
+ \nu(\nabla \cdot \mathbf{v}, \nabla \cdot \tilde{v}) - \nu \int_{\partial \mathcal{M}} (\nabla \cdot \mathbf{v})(\tilde{v} \cdot n) dS.
\]

(6.13)

Since \(\tilde{v} \in V\), we need only assume that \(\text{curl}(\mathbf{v}) = 0\) on \(\partial \mathcal{M}\) to get rid of the boundary terms. With this assumption, we see that

\[
-\nu(\Delta \mathbf{v}, \tilde{v}) = \nu((\mathbf{v}, \tilde{v})),
\]

(6.14)

where \((\cdot, \cdot)\) denotes the inner product on \(V\). For the nonlinear term in (6.12), we obtain:

\[
((\mathbf{v} \cdot \nabla) \mathbf{v}, \tilde{v}) = \frac{1}{2} (\nabla \mathbf{v}^2, \tilde{v}) + (\text{curl}(\mathbf{v}) \alpha(\mathbf{v}), \tilde{v})
\]

\[
= -\frac{1}{2} (\mathbf{v}^2, \nabla \cdot \tilde{v}) + (\text{curl}(\mathbf{v}) \alpha(\mathbf{v}), \tilde{v}) + \frac{1}{2} \int_{\partial \mathcal{M}} |\mathbf{v}|^2 (\tilde{v} \cdot n) dS,
\]

(6.15)

where again the boundary term vanishes since \(\tilde{v} \in V\). Another application of the divergence theorem yields:

\[
g(\nabla h, \tilde{v}) = -g(h, \nabla \cdot \tilde{v}) + g \int_{\partial \mathcal{M}} h(\tilde{v} \cdot n) dS.
\]

(6.16)

Now we take (1.1b), multiply it by \(\tilde{h} \in V_2\), and integrate over \(\mathcal{M}\):

\[
\left( \frac{\partial h}{\partial t}, \tilde{h} \right) + (\nabla \cdot (h \mathbf{v}), \tilde{h}) - \delta(\Delta h, \tilde{h}) = 0.
\]

(6.17)

The second term can be rewritten as:

\[
(\nabla \cdot (h \mathbf{v}), \tilde{h}) = -(h \mathbf{v}, \nabla \tilde{h}) + \int_{\partial \mathcal{M}} h \tilde{h}(\mathbf{v} \cdot n) dS.
\]

(6.18)

If we assume that \(\mathbf{v} \cdot n = 0\) on \(\partial \mathcal{M}\), then the boundary term vanishes. For the third term in (6.17), we have

\[
-\delta(\Delta h, \tilde{h}) = \delta((h, \tilde{h})) - \delta \int_{\partial \mathcal{M}} \tilde{h}(\nabla h \cdot n) dS = \delta((h, \tilde{h})) - \delta \int_{\partial \mathcal{M}} \tilde{h}(\nabla h \cdot n) dS.
\]

(6.19)
To get rid of the boundary term, we must assume that $\nabla h \cdot n = 0$ or $\tilde{h} = 0$ on $\partial M$. We see that with these boundary conditions (or Dirichlet boundary conditions on $v$ and $h$):

$$
\begin{align*}
v \cdot n &= 0 \quad \text{on } \partial M \times (0, T), \\
\text{curl}(v) &= 0 \quad \text{on } \partial M \times (0, T), \\
\text{and } \nabla h \cdot n &= 0 \quad \text{on } \partial M \times (0, T),
\end{align*}
$$

the original problem is equivalent to the variational problem:

Find $v \in L^2(0, T; V) \cap L^\infty(0, T; H)$, $h \in L^2(0, T; V_2) \cap L^\infty(0, T; H_2)$ s.t.

$$
\begin{align*}
\left( \frac{\partial v}{\partial t}, \tilde{v} \right) + \nu((v, \tilde{v})) - \frac{1}{2}(v^2, \nabla \cdot \tilde{v}) + (\text{curl}(v) \alpha(v), \tilde{v}) - g(h, \nabla \cdot \tilde{v}) + (f k \times v, \tilde{v}) &= (F, \tilde{v}) \quad \forall \tilde{v} \in V,
\end{align*}
$$

(6.20)

$$
\begin{align*}
\left( \frac{\partial h}{\partial t}, \tilde{h} \right) - (hv, \nabla h) + \delta((h, \tilde{h})) &= 0 \quad \forall \tilde{h} \in V_2,
\end{align*}
$$

(6.21)

$$
\begin{align*}
v(t = 0) &= v^0(x, y), \quad \text{and } h(t = 0) = h^0(x, y) > 0 \quad \text{in } M.
\end{align*}
$$

(6.22)

6.4. Appendix D

In this section, we prove the positivity of $h$. Recall that $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$ is the total derivative of $f(x, y, t)$. Then we can rewrite (1.1b) as

$$
\frac{d\ln(h)}{dt} = -\nabla \cdot v + \frac{\delta}{h} \Delta v + \frac{1}{h} \sigma_2(v, h) \frac{dW_2}{dt},
$$

(6.23)

along the characteristics $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$. Hence,

$$
h(t) = h_0 \cdot \exp \left\{ \int_0^t -\nabla \cdot v + \frac{\delta}{h} \Delta v ds + \int_0^t \frac{1}{h} \sigma_2(v, h) dW_2 \right\}.
$$

(6.24)

If we assume that $v \in C^1$, then the characteristics cover all of $M$ and we see that the exponent above is bounded. This implies that $h$ is strictly positive due to the positivity assumption on $h_0$ in (1.1d).

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References


