1. Fiat Currency in an Arrow-Debreu Economy

In an Arrow-Debreu economy, with complete contingent claims markets, pure fiat currency has no value. Fiat currency is defined to be intrinsically useless, meaning that it does not directly enter into utility or production functions and private agents are not forced to hold it.

To see that fiat currency, $M$, has no value, consider a representative household that chooses sequences $\{c_t, s_{t+1}, m_{t+1}\}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1,$$

subject to

$$c_t + q_t s_{t+1} + \omega_t m_{t+1} \leq s_t (q_t + y_t) + \omega t m_t,$$

where $q_t$ is the price of trees, $s_t$ is the number of trees owned at the beginning of $t$, $\omega_t$ is the value of currency at $t$ (inverse of the general price level), $m_t$ is the amount of currency owned at the beginning of $t$, and $y_t > 0$, all $t$, is the non-random dividend each tree yields at $t$. The utility function, $u$, is strictly increasing, strictly concave, and twice continuously differentiable. Assume that there are $M$ units of currency available in the economy each period. If you find it convenient, assume that $y_t = y$, all $t$. We shall prove that in equilibrium $\omega_t = 0$, $t \geq 0$.

The first-order conditions imply

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} \left( \frac{q_{t+1} + y_{t+1}}{q_t} \right) = 1,$$

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} \omega_{t+1} \leq \omega_t \quad = \text{if } m_{t+1} > 0.$$

Equilibrium requires that $c_t = y_t$ and $m_t = M$ for all $t \geq 0$. Imposing equilibrium on (3) yields the usual asset-pricing equation, $q_t = \beta(q_{t+1} + y)$, when $y_t = y$ for all $t$. 

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To show that fiat currency has no value, we shall assume the contrary and then show that this leads to a contradiction. Assume there is an equilibrium in which $\omega_t > 0, t \geq 0$. Since in equilibrium, $m_t = M$, (4) holds with equality, implying that

$$\omega_{t+1} = \beta^{-1}\omega_t,$$

where we have used that $y_t = y$, all $t$. If $p_0 > 0$, then (5) implies that $\omega_t \to \infty$ as $t \to \infty$, so the value of money is unbounded. But then the consumer’s initial wealth at $t$, $s_t(q_t + y_t) + \omega_t m_t$, grows without bound. In this case, the allocation $c_t = y_t$ cannot be optimal because it would be feasible to consume more in some periods without consuming less in other periods. Hence, such a path for $\{\omega_t\}$ cannot be an equilibrium, and we cannot have $p_0 > 0$. Instead, we must have $p_0 = 0$, so $\omega_t = 0$ for $t \geq 1$. Notice that $\omega_t = 0$ for $t \geq 0$ satisfies (4). In this equilibrium, fiat currency has no value.

As a consequence of this general result, one branch of monetary economics uses a variety of ad hoc methods for introducing fiat currency and getting it valued in general equilibrium models. We now explore several of these methods.

2. Money-in-the-Utility-Function

Assume that real balances, $m_t/p_t$, enter utility directly.

This is justified by acknowledging that agents must “shop” to acquire goods and shopping takes time. Let $l_p$ denote time spent shopping to consume $c$ units of goods. Agents value leisure according to $\phi(1 - l_p), \phi' > 0$, where we have assumed a time endowment of unity each period. In an economy without currency agents must barter, so $1 - l_p$ is lower. Write $1 - l_p = L(x_t)$, where $x_t = m_t/p_t$, with $L' > 0$. Then the utility of leisure is

$$\phi(1 - l_p) = \phi(L(x_t)).$$

Since $\phi$ and $L$ are both increasing functions, the composite function $\phi(L(\cdot))$ is increasing. Denote the composite function by $v$, so $v(x_t)$ is the utility of leisure and $v' > 0$. So money-in-the-utility-function (MIUF) is a short-cut (or reduced-form) way of saying that currency reduces transactions time.

Assume that $v(m_t/p_t)$ with $v' > 0, v'' < 0$.

Suppose households solve

$$\max_{\{c_t,s_{t+1},m_t,b_t\}} \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \gamma \log(m_t/p_t)],$$

subject to
\[ ct + \tau t + atst_{t+1} + qtbt + \frac{mt}{pt} \leq (at + yt)st + b_{t-1} + \frac{m_{t-1}}{pt}, \]  
(8)

with \( ct \geq 0, m_t \geq 0, \) all \( t, \) given \( s_0 = 1, b_{-1} = 0, m_{-1} > 0. \) Competitive agents take \( \{at, qt, pt\}_{t=0}^{\infty} \) and government policies as given.

The first-order condition for \( m \) (for an interior solution) is

\[ \frac{\lambda_t}{pt} = \frac{\gamma}{m_t} + \beta \frac{\lambda_{t+1}}{pt_{t+1}}, \]  
(9)

where \( \lambda \) is the lagrange multiplier on (8). With log preference over \( c, \) (9) is

\[ \beta \frac{c_t p_t}{c_{t+1} pt_{t+1}} + \gamma \frac{c_t pt}{m_t} = 1. \]  
(10)

There are two immediate implications of the optimality condition:

1. Bonds dominate money in rate of return.

\[ qt = \beta \frac{c_t}{c_{t+1}} = \frac{1}{R_t}, \]  
(11)

and (10) implies

\[ \frac{pt_{t+1}}{R_t} = 1 - \frac{pt c_t}{m_t}. \]  
(12)

so if money is valued, \( m_t/pt > 0, \) which implies \( pt/pt_{t+1} < R_t. \)

2. Return dominance is necessary (and sufficient) for there being positive demand for money.

The demand for money is given by

\[ \frac{m_t}{pt} = \frac{\gamma c_t}{1 - \frac{pt_{t+1}}{R_t}} > 0 \text{ iff } pt/pt_{t+1} < R_t. \]  
(13)

The bonds in (8) are indexed (real) bonds. A nominal bond can also be priced. It sells for \( Qt \) units of money at \( t \) and promises to pay one unit of currency at \( t + 1. \) Let \( i_t \) be the (net) nominal interest rate, so \( 1 + i_t = 1/Qt. \) The marginal condition for pricing this asset is

\[ 1 + i_t = \frac{1}{Qt} = \frac{\beta w'(ct)}{\beta w'(ct_{t+1})} \frac{pt_{t+1}}{pt} = R_t \frac{pt_{t+1}}{pt}, \]  
(14)

which is the Fisher equation for this model. Note that currency is valued if and only if \( i_t > 0. \)

The demand for money, (13) can now be expressed in a more conventional form as
\[
\frac{m_t}{p_t} = \gamma c_t \left( 1 + \frac{1}{\mu_t} \right).
\] (15)

Money demand is increasing in the level of transactions, \(c\), and decreasing in the opportunity cost, \(i\).

To derive the equilibrium, we need to specify the behavior of government policy. For simplicity, suppose government consumption and government debt are zero and the government merely returns seigniorage revenues to the household as a lump-sum transfer. Then the government budget constraint is

\[
-\tau_t = \frac{M_t - M_{t-1}}{p_t},
\] (16)

where \(\tau\) is lump-sum taxes (if positive) and transfers (if negative), while \(M_t\) is the aggregate money stock at \(t\).

Equilibrium requires that markets clear:

\[
c_t = y_t
\]
\[
b_t = 0
\]
\[
s_{t+1} = 1
\]
\[
m_t = M_t.
\]

Let \(z_t = 1/p_t c_t\) and note that (10) is a first-order difference equation in \(z\):

\[
z_t = \beta z_{t+1} + \frac{\gamma}{m_t},
\] (17)

(17) is often called a “Cagan money demand.” So long as \(\lim_{i \to \infty} \beta^i z_{t+i} = 0\), the solution to (17) is

\[
\frac{1}{p_t} = \gamma y_t \sum_{t=0}^{\infty} \beta^t \frac{1}{M_{t+i}},
\] (18)

or

\[
\frac{M_t}{p_t} = \gamma y_t \mu_t,
\] (19)

where

\[
\mu_t \equiv \sum_{i=0}^{\infty} \beta^i \prod_{j=0}^{i-1} \frac{1}{\rho_{t+j+1}},
\] (20)

with \(\rho_t \equiv M_t/M_{t-1}\) is the money growth rate.
Some features of the equilibrium include:

1. Money is neutral and super-neutral. \( c_t = y_t \) is independent of monetary policy, as are real asset prices.
2. \( p_t \) depends on all future money stocks, so higher expected money raises the current price level.
3. Real balances depend on all expected growth rates of money. These are the rates of depreciation in the value of money, so \( \mu_t \) can be interpreted as the present value of real money balances.

3. Cash-in-Advance

We now add money to a standard growth model with inelastic physical capital stock. We seek to do this in a way that creates the least distortion so that the underlying real model emerges under certain conditions.

Household preferences are given by

\[
\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \quad 0 < \beta < 1, \tag{21}
\]

where \( n \) is labor, not leisure. The aggregate resource constraint is

\[
c_t = n_t, \tag{22}
\]

which assumes that production is linear in labor.

We assume that money is fundamental to the activity of exchanging goods. Goods can be exchanged for money, but goods cannot be exchanged for goods. This asymmetry is essential, as it ascribes to money a unique role. The household is composed of two members—a shopper and a worker. Shoppers and workers carry out non-synchronized activities.

We sub-divide each period into two parts. First the goods market opens. A household starts with an amount of money, \( m_{t-1} \), and receives transfers from the government, \( \tau_t \). The shopper goes to the goods market and buys goods with money. This creates the cash-in-advance constraint:

\[
m_{t-1} + \tau_t \geq p_t c_t. \tag{23}
\]

The worker rents labor to the firm at rate \( w \). Wages cannot be paid until the firm has sold its goods, and by that time the goods market is closed. This means there is a synchronization problems facing the household since it cannot simply convert its wages at \( t \) into consumption goods at \( t \). The firm’s problem is to choose \( n_t \) to maximize
After the goods market closes, asset markets open. The household receives payment for its labor. Assets contracted for in the previous period pay off. All payoffs are in nominal terms and these nominal payoffs are used to buy new assets and money. Assume for now there are only one-period pure discount bonds. Total nominal receipts of the household are

\[ w_t n_t + b_{t-1} + (m_{t-1} + \tau_t - p_t c_t). \]  

Total household expenditures on new assets and money are

\[ q_t b_t + m_t. \]  

The household also faces a no-Ponzi scheme constraint, which puts a lower bound in its debt:

\[ \frac{b_t}{p_t} \geq -B. \]  

Pulling it all together, the household’s budget constraint is

\[ w_t n_t + b_{t-1} + (m_{t-1} + \tau_t - p_t c_t) \geq q_t b_t + m_t. \]  

The government in this economy announces a sequence of monetary transfers, \( \{\tau_t\} \). The aggregate money stock evolves as

\[ M_t = M_{t-1} + \tau_t, \]  

where \( M_{t-1} \) is the per capita level of money at the beginning of \( t \). We assume bonds are traded only among private agents, implying both that they are in zero net supply and that (29) serves as the government budget constraint.

The household chooses sequences \( \{c_t, n_t, b_t, m_t\} \) to maximize (21) subject to (23), (28), and (27). Letting \( \mu, \lambda, \) and \( \eta \) be the lagrange multipliers on (23), (28), and (27), a solution to the household’s problem satisfies

\[ u_c(t) - (\mu_t + \lambda_t) = 0, \]  

\[ u_n(t) + \frac{w_t}{p_t} \lambda_t = 0, \]  

\[ -\frac{\lambda_t}{p_t} + \beta \left( \frac{\lambda_{t+1} + \mu_{t+1}}{p_{t+1}} \right) = 0, \]
\[-q_t \frac{\lambda_t}{p_t} + \beta \frac{\lambda_{t+1}}{p_{t+1}} = 0, \tag{33}\]

and the transversality conditions

\[
\lim_{t \to \infty} \beta^t \lambda_t \left( \frac{b_t}{p_t} + B \right) = 0, \tag{34}\]

\[
\lim_{t \to \infty} \beta^t \lambda_t \frac{m_t}{p_t} = 0. \tag{35}\]

The firm’s problem implies that

\[
w_t = p_t. \tag{36}\]

Because in equilibrium \( b_t = 0, \eta_t = 0 \) for all \( t \).

In this economy, no one would hold money if they were not required to use money to acquire goods. This is because money bears a positive opportunity cost and goods are costless to hold. One can also think of this in terms of interest foregone by holding money instead of bonds.

To understand the nature of the distortion money creates, consider a social planner’s problem, which finds efficient allocations subject to constraints, including those imposed by the existence of money and the prohibition of goods-for-goods exchange.

For simplicity, we dispense with the firm, so we set \( w_t = p_t \) and we dispense with debt, since \( b_t = 0 \).

The planner’s problem is to choose \( \{c_t, n_t, p_t\} \) to maximize

\[
\sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{37}\]

subject to

\[
m_{t-1} + \tau_t \geq p_t c_t, \tag{38}\]

\[
p_t n_t + (m_{t-1} + \tau_t - p_t c_t) \geq m_t \tag{39}\]

\[
n_t = c_t, \tag{40}\]

with \( m_{-1} = M_{-1} \) given.

Note that it is always possible to set \( p \) so that the cash-in-advance constraint does not bind. Because \( c_t = n_t \), one can set \( m_t = m_{t-1} + \tau_t \) so all money is held. Hence,
including the cash-in-advance constraint and the budget constraint does not prevent us from choosing efficient levels of $c$ and $n$.

Consider first constant money growth rates. Assume

$$\tau_t = (\rho - 1) M_{t-1}. \quad (41)$$

We claim that if $M_t = \rho M_{t-1}$, then as long as $\rho > \beta$, $-u_n < u_c$, so the allocation is inefficient. We prove this by assuming that $\rho > \beta$ and $-u_n = u_c$. The complementary slackness condition associated with the cash-in-advance constraint is

$$\frac{m_{t-1}}{p_t} + \tau_t - c_t \leq 0, \quad \mu_t \left[ \frac{m_{t-1}}{p_t} + \tau_t - c_t \right] = 0. \quad (42)$$

In a monetary equilibrium, $\mu_t > 0$, so in equilibrium, $c_t = M_t/p_t$. But the first-order conditions imply

$$\frac{\lambda_t}{p_t} = \beta \frac{u_c(t+1)}{p_{t+1}} = -\frac{u_n(t)}{p_t}, \quad (43)$$

so if $-u_n = u_c$, the $\mu_t = 0$. Combining $-u_n(t) = u_c(t)$ with the aggregate resource constraint that $c_t = n_t$ implies that $-u_n(c_t, c_t) = u_c(c_t, c_t)$, which produces unique, bounded (in fact, constant) sequences for $\{c_t, n_t\}$. But $c_t = c$ and $n_t = n$ implies

$$p_{t+1} = \beta p_t, \quad (44)$$

while $M_{t+1} = \rho M_t$. By (42), $\mu_t = 0$ implies $M_t/p_t \leq c$. So we have that

$$\frac{M_{t+1}}{p_{t+1}} = \frac{\rho M_t}{\beta p_t}, \quad (45)$$

which implies that $M_t/p_t \to \infty$, contradicting the requirement that $M_t/p_t \leq c$. The basic economic reason for this proposition is that if $\rho > \beta$, then agents would like to hold zero money balances, but they cannot if consumption is to be positive.

To construct an equilibrium when $\rho \geq \beta$, conjecture there is a stationary allocation where the cash-in-advance constraint binds. Then

$$p_t = \frac{\rho^t M_0}{c}, \quad (46)$$

where $c$ is the stationary level of consumption. Then we have that

$$-u_n = \lambda \quad (47)$$

$$u_c = \mu + \lambda, \quad (48)$$
where $\mu$ is the wedge that creates the inefficiency in the monetary equilibrium. Then because in a stationary equilibrium, $p_t/p_{t+1} = 1/\rho$,

$$-\lambda + \beta \left( \frac{\mu + \lambda}{\rho} \right) = 0. \quad (49)$$

Finally,

$$q = \frac{\beta}{\rho}. \quad (50)$$

Collecting equilibrium conditions, $(c, n, \lambda, \mu)$ are determined by

$$u_c = \mu + \lambda$$
$$-u_n = \lambda$$
$$\lambda = \frac{\beta}{\rho} (\mu + \lambda)$$
$$c = n.$$

All that remains is to check that the transversality condition for money is satisfied:\footnote{Because we assumed $b = 0$, the transversality condition for debt is satisfied trivially.}

$$\lim_{t \to \infty} \beta^t \frac{\lambda^t M_t}{p_t} = \lim_{t \to \infty} \beta^t \frac{-u_n^t \rho^t M_0}{\rho^t M_0/c} = \lim_{t \to \infty} \beta^t \frac{-u_n}{1/c} = 0. \quad (51)$$

In this economy, inflation works like a wage tax. Money drives a wedge between $u_n$ and $u_c$, just as a tax on labor does:

$$-\frac{u_n}{u_c} = \frac{\lambda}{\mu + \lambda} = (1 - \tau_n). \quad (52)$$

Labor earnings are being taxed by the delay before they can be used. Inflation over the period erodes their real value before they can be spent. The extent to which positive money growth is distorting the labor-consumption margin is increasing in $\rho$ because

$$-u_n = \beta u_c / \rho. \quad (53)$$

The first-order conditions imply

$$q_t = \frac{\lambda_{t+1}}{\mu_{t+1} + \lambda_{t+1}}. \quad (54)$$

so $q$, the price of bonds, measures the expected distortion. This expected distortion is zero if and only if $q = 1$, so $i = 0.$
This leads to the conclusion that an efficient outcome occurs if ρ = β, so the price level is falling at rate β and q = 1. This is the Friedman rule. Because the nominal interest rate is the cost of holding money, and money is a good that is socially costless to produce, people should be satiated in it and its price should be zero.

We now turn to consider stochastic money growth. Assume that the growth rate of money is i.i.d. and the lower support of ρ is greater than β and the upper support is finite. Let  \( \bar{\rho} = E[\rho] \) and \( \tau_t = (\rho_t - 1)M_{t-1} \), as before. The household’s problem is now

\[
L_t = \max_{\{c_t, n_t, b_t, m_t\}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, n_t) + \mu_t \left[ \frac{m_t-1+\tau_t}{p_t} - c_t \right] + \lambda_t \left[ n_t + \frac{b_{t-1}}{p_t} + \left( \frac{m_{t-1}+\tau_{t-1}}{p_t} - c_t \right) - q_t \frac{b_t}{p_t} - \frac{m_t}{p_t} \right] + \eta_t \frac{b_t}{p_t} + \bar{B} \right\}.
\] (55)

The first-order conditions can be reduced to finding three functions, \( c(\rho_t) \), \( n(\rho_t) \), \( p(\rho_t) \) that when the cash-in-advance constraint holds with equality satisfy

\[
c(\rho_t) = n(\rho_t),
\] (56)

\[
c(\rho_t) = \rho_t \frac{M_{t-1}}{p_t},
\] (57)

\[
-\frac{u_n(\rho_t)}{u_c(\rho_t)} = \beta E_t \frac{u_c(\rho_{t+1})}{p(\rho_{t+1})}.
\] (58)

We can reduce these three equations to

\[
-\frac{u_n(n(\rho_t), n(\rho_t))}{u_c(\rho_t)} n(\rho_t) = \beta E_t \left[ u_c(n(\rho_{t+1}), n(\rho_{t+1})) \left( \frac{n(\rho_{t+1})}{\rho_{t+1}} \right) \right].
\] (59)

Note that in (59), only \( \rho_{t+1} \) appears, and then only as an expectation. When money shocks are i.i.d., the current realization of money growth has no real effects, so unanticipated money is neutral while anticipated money is non-neutral. One can also show that in steady state

\[
\frac{\partial \bar{n}}{\partial \bar{\rho}} < 0,
\] (60)

where \( \bar{n} \) and \( \bar{\rho} \) denote steady state values.

To solve (59), linearize around \( (\bar{n}, \bar{\rho}) \) to obtain
\[-[u_{nn}(\bar{n}, \bar{n}) + u_{nc}(\bar{n}, \bar{n}) + u_n(\bar{n}, \bar{n})]\bar{n}_t\]

\[= \beta E_t \left\{ \left[ u_{cc}(\bar{n}, \bar{n}) \frac{\bar{n}}{\bar{\rho}} + u_{cn}(\bar{n}, \bar{n}) \frac{\bar{n}}{\bar{\rho}} + u_c(\bar{n}, \bar{n}) \frac{1}{\bar{\rho}} \right] \bar{n}_{t+1} - u_c(\bar{n}, \bar{n}) \frac{\bar{n}}{\bar{\rho}^2} \bar{\rho}_{t+1} \right\},\]

where \(\tilde{x}_t = x_t - \bar{x}\) denotes deviation from steady state. Write (61) as

\[A\tilde{n}_t = E_t[B\tilde{n}_{t+1} + C\tilde{\rho}_{t+1}].\]  

Suppose that \(\tilde{\rho}_{t+1} = \phi \tilde{\rho}_t + \varepsilon_{t+1}\). We conjecture that \(\tilde{n}_t = DE_t\tilde{\rho}_{t+1}\), so \(\tilde{n}_{t+1} = D\phi^2\tilde{\rho}_t\). Substitute into (62) to get

\[AD\phi\tilde{\rho}_t = BD\phi^2\tilde{\rho}_t + C\phi\tilde{\rho}_t,\]

so \(D = C\phi/(A\phi - B\phi^2)\) and we obtain the decision rule

\[\tilde{n}_t = D\phi\tilde{\rho}_t,\]  

where \(D < 0\). Whenever money growth is positively serially correlated, higher current money growth predicts high future money growth, which reduces employment.

### 4. Cash and Credit Goods

While cash-in-advance has the advantage over money-in-the-utility function in being explicit about the exchanges that take place, it has the disadvantage that in equilibrium velocity is constant. Here we briefly present an alternative cash-in-advance specification, due to Lucas and Stokey (1987), that addresses this shortcoming of the simple CIA model. We focus on a simple parametric example.

There are two kinds of consumption goods—those that are purchased via a CIA constraint (“cash goods”) and those that can be bought without money (“credit goods”). Output is produced with labor, with capital fixed, using linear technologies:

\[y_{1t} = h_{1t}\]  

\[y_{2t} = h_{2t},\]  

where \(h_{it}\) is hours worked at \(t\) producing good \(i\). The aggregate resource constraints are

\[c_{1t} = y_{1t},\]  

\[c_{2t} = y_{2t}.\]
Good 1 is the “cash good” and good 2 is the “credit good.”

Again, we imagine the household consisting of a worker/shopper pair. The shopper buys the cash good subject to the CIA constraint

\[ m_{t-1} + \tau_t + b_{t-1} - b_t/R_t \geq p_t c_{1t}, \]  

(68)

where, in contrast to the previous model, net holdings of nominal bonds may serve as a source of liquidity, but, as before, lump-sum taxes/transfers, \( \tau_t \), enter the CIA constraint. The household’s budget constraint is

\[ m_{t-1} + \tau_t + b_{t-1} + p_t(h_{1t} + h_{2t}) \geq p_t(c_{1t} + c_{2t}) + b_t/R_t + m_t. \]  

(69)

The household chooses sequences \( \{c_{1t}, c_{2t}, h_{1t}, h_{2t}, b_t, m_t\} \) to maximize

\[
\sum_{t=0}^\infty \beta^t [\alpha \log(c_{1t}) + (1 - \alpha) \log(c_{2t}) - \gamma(h_{1t} + h_{2t})] \]

(70)

subject to (68) and (69).

We focus on an equilibrium in which \( b_t = b_{t-1} = 0 \). In addition to the first-order conditions, an equilibrium must satisfy

\[ c_{1t} = \frac{M_{t-1} + \tau_t}{p_t}, \]  

(71)

\[ M_{t-1} + \tau_t = M_t, \]  

(72)

\[ \frac{M_t}{p_t} = c_{1t}, \]  

(73)

plus (64)-(67). Note that (73) implies that velocity, measured in terms of good 1 consumption, is constant, just as in the previous CIA model. The first-order conditions imply

\[ c_{1t} = h_{1t} = \frac{\alpha \beta p_{t-1}}{\gamma / p_t}, \]  

(74)

\[ c_{2t} = h_{2t} = \frac{1 - \alpha}{\gamma}, \]  

(75)

\[ R_t = \frac{p_{t+1}}{\beta p_t} \left( \frac{\gamma + \mu_t}{\gamma + \mu_{t+1}} \right) \]

\[ = \frac{1}{\beta} \frac{p_t}{p_{t-1}}. \]  

(76)
where \( \mu \) is the multiplier associated with the CIA constraint.

Combining (73), (74), and (76) yields equilibrium real money balances (some times called “money demand”):

\[
\frac{M_t}{p_t} = \frac{\alpha}{\gamma R_t}. \tag{77}
\]

We can summarize the results as:

1. Neutrality of money. Suppose for a given monetary policy, \( \{M_t\}_{t=0}^\infty \), an equilibrium is given by \( \{c_{1t}, c_{2t}, h_{1t}, h_{2t}, p_t, R_t\}_{t=0}^\infty \). Then for any alternative monetary policy, \( \{\delta M_t\}_{t=0}^\infty, \delta > 0, \{c_{1t}, c_{2t}, h_{1t}, h_{2t}, \delta p_t, R_t\}_{t=0}^\infty \) is an equilibrium.

2. If \( \alpha = 0 \), the demand for money is zero and money is not valued.

3. The interest elasticity of money demand is negative, so if \( R \) rises, households consume less of the cash good.

4. A Fisher equation holds. From (76), \( R_t = \beta^{-1} p_t / p_{t-1} \), so inflation changes the nominal interest rate proportionally.

5. Money is not superneutral. A permanent increase in money growth, \( \rho \), increases inflation and the nominal interest rate. The household economizes on holding money balances by consuming less of the cash good. This reduces total output.

To see (5), let \( M_t / M_{t-1} = \rho \) denote a constant growth rate of the money supply. We seek a steady state with constant real money balances so that \( (M_t / p_t) / (M_{t-1} / p_{t-1}) = 1 \). Then \( p_t / p_{t-1} = \rho \). From (74),

\[
c_{1t} = h_{1t} = \frac{\alpha \beta}{\gamma} \frac{1}{\rho}, \tag{78}
\]

and aggregate output is

\[
y_t = c_{1t} + c_{2t} = \frac{\alpha \beta}{\gamma} \frac{1}{\rho} + \frac{1 - \alpha}{\gamma}. \tag{79}
\]

(6) The optimum quantity of money obeys the Friedman rule: \( \rho = \beta \).

To show (6), we find the \( \rho \) that maximizes household utility. Note that in a steady state, \( R_t = \rho / \beta \), so that \( c_{1t} = M_t / p_t = \alpha \beta / \gamma \rho \). Substitute this and previous results into (70) to obtain

\[
\sum_{t=0}^\infty \beta^t \left[ \alpha \log \left( \frac{\alpha \beta}{\gamma \rho} \right) + (1 - \alpha) \log \left( \frac{1 - \alpha}{\gamma} \right) - \gamma \left( \frac{\alpha \beta}{\gamma \rho} + \frac{1 - \alpha}{\gamma} \right) \right]. \tag{80}
\]

The first-order condition with respect to \( \rho \) is
\[ 0 = \sum_{t=0}^{\infty} \beta^t \left[ -\frac{\alpha}{\rho} + \frac{\alpha \beta}{\rho^2} \right]. \tag{81} \]

By inspection of (81), \( \rho = \beta \), so \( R = \frac{\rho}{\beta} = 1 \).

References