This note considers optimal monetary policy behavior in the simplest forward-looking version of the popular class of DSGE models with nominal rigidities. Woodford (2003) exhaustively examines many variants on this model. An important variant arises when both prices and wages are sticky; Erceg, Henderson, and Levin (2000) consider this variant. We will derive optimal policy both with commitment and without commitment.

We begin with the canonical New Keynesian model described by an “IS” equation
\[ x_t = E_t x_{t+1} - \sigma^{-1}(i_t - E_{t+1}) + u_t \] (1)
and an “AS” equation
\[ \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + e_t. \] (2)
All parameters are positive and the shocks are independent. The central bank has the loss function
\[ L_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{1}{2} \left( \pi_{t+j}^2 + \lambda x_{t+j}^2 \right), \] (3)
and seeks to maximize (3) subject to (1) and (2). Formulate the lagrangian
\[ E_t \sum_{j=0}^{\infty} \beta^j \left\{ \frac{1}{2} \left( \pi_{t+j}^2 + \lambda x_{t+j}^2 \right) + \theta_{t+j} \left[ x_{t+j} - x_{t+j+1} + \sigma^{-1}(i_{t+j} - \pi_{t+j+1}) - u_{t+j} \right] + \psi_{t+j} \left[ \pi_{t+j} - \beta \pi_{t+j+1} - \kappa x_{t+j} - e_{t+j} \right] \right\}, \] (4)
and take first-order conditions with respect to \( i_{t+j}, \pi_{t+j}, x_{t+j}, j \geq 0 \).

1. Optimal Policy with Commitment
First note that the first-order condition with respect to \( i_{t+j} \) implies
\[ \sigma^{-1} E_t \theta_{t+j} = 0, \] (5)
which implies that (1) imposes no constraints on monetary policy as long as there are no costs to varying \( i \). (This would change if the central bank’s loss function penalized changes in the interest rate, as many models of interest-rate smoothing assume.) An immediate implication of (5) is that given optimal choices for \( \pi \) and \( x \), (1) simply gives the \( i \) required to implement the optimal policy. Hence, one could pose the optimal policy problem instead as choosing \( \{\pi_{t+j}, x_{t+j}\} \) to minimize (3) subject to (2), treating \( x \) as the policy instrument.

We now compute the first-order conditions for \( \pi \) and \( x \) for \( j = 0 \) and \( j \geq 1 \):

\[
\begin{align*}
E_t(\pi_t + \psi_t) &= 0, \quad (6) \\
E_t(\lambda x_t - \kappa \psi_t) &= 0, \quad (7) \\
E_t \left[ \beta^j (\pi_{t+j} + \psi_{t+j}) - \beta^{j-1}(\sigma^{-1}\theta_{t+j-1} + \beta \psi_{t+j-1}) \right] &= 0 \quad \text{(8)} \\
E_t[\beta^j (\lambda x_{t+j} + \theta_{t+j} - \kappa \psi_{t+j}) - \beta^{j-1}\theta_{t+j-1}] &= 0. \quad (9)
\end{align*}
\]

Taken together, these imply

\[
\begin{align*}
\pi_t + \psi_t &= 0, \quad (10) \\
E_t(\pi_{t+j} + \psi_{t+j} - \psi_{t+j-1}) &= 0, \quad j \geq 1, \quad (11) \\
E_t(\lambda x_{t+j} - \kappa \psi_{t+j}) &= 0, \quad j \geq 0. \quad (12)
\end{align*}
\]

Conditions (10) and (11) reveal the dynamic inconsistency problem. At \( t \), the central bank obeys (10) and sets \( \pi_t = -\psi_t \) and then promises to set \( \pi_{t+1} = -(\psi_{t+1} - \psi_t) \), according to (11). But at \( t + 1 \), if the central bank were to reoptimize, it would be optimal to obey (10) and set \( \pi_{t+1} = -\psi_{t+1} \). The timeless perspective essentially implements (11) and (12) for all \( j \), including \( j = 0 \), as a means of getting around the inconsistency problem.

Conditions (11) and (12) imply

\[
\pi_{t+j} = -\frac{\lambda}{\kappa}(x_{t+j} - x_{t+j-1}), \quad j \geq 0. \quad (13)
\]

Using (13) in (2) produces a difference equation in \( x \):
\[
\left(1 + \beta + \frac{\kappa^2}{\lambda}\right)x_t = \beta E_t x_{t+1} + x_{t-1} - \frac{\kappa}{\lambda} e_t. \tag{14}
\]

Posit a solution that is a function of the state \((x_{t-1}, e_t)\):

\[
x_t = a_x x_{t-1} + b_x e_t, \tag{15}
\]

and assume the shock is AR(1):

\[
e_t = \rho e_{t-1} + \epsilon_t. \tag{16}
\]

Then

\[
E_t x_{t+1} = a_x x_t + b_x \rho e_t = a_x^2 x_{t-1} + b_x (a_x + \rho) e_t. \tag{17}
\]

Substituting (15) and (17) into (14) yields

\[
\left(1 + \beta + \frac{\kappa^2}{\lambda}\right) (a_x x_{t-1} + b_x e_t) = (1 + \beta a_x^2) x_{t-1} + \left[\beta b_x (a_x + \rho) - \frac{\kappa}{\lambda}\right] e_t. \tag{18}
\]

Matching coefficients in (18),

\[
a_x \left(1 + \beta + \frac{\kappa^2}{\lambda}\right) = 1 + \beta a_x^2
\]

\[
b_x \left(1 + \beta + \frac{\kappa^2}{\lambda}\right) = \beta b_x (a_x + \rho) - \frac{\kappa}{\lambda}
\]

A stationary solution has \(|a_x| < 1\) satisfying

\[
\beta a_x^2 - a_x \left(1 + \beta + \frac{\kappa^2}{\lambda}\right) + 1 = 0
\]

and

\[
b_x = -\left[\frac{\kappa}{\lambda[1 + \beta(1 - \rho - a_x)] + \kappa^2}\right].
\]

The resulting decision rule for \(\pi\) is

\[
\pi_t = \left(\frac{\lambda}{\kappa}\right) (1 - a_x) x_{t-1} + \left[\frac{\lambda}{\lambda[1 + \beta(1 - \rho - a_x)] + \kappa^2}\right] e_t. \tag{19}
\]

One can now solve for the reduced-form expression for the nominal interest rate. Use (1) to solve for \(i_t\), use (15) and (17) for \(x_t\) and \(E_t x_{t+1}\), and use (19) to find \(E_t \pi_{t+1}\). The result is
\[ i_t = a_x \left[ \sigma(a_x - 1) + \frac{\lambda}{\kappa}(1 - a_x) \right] x_{t-1} + b_x \left[ \frac{\lambda}{\kappa}(1 - a_x) - \sigma(1 - a_x - \rho) \right] e_t + \sigma u_t. \]  

(20) is the timeless perspective solution for \( i_t \) as a function of the state.

2. **Optimal Policy with Discretion**

Under discretion, the central minimizes (3) subject to (2), as before. The difference is that the central bank’s choices at \( t \) do not bind in the future, so the central bank cannot affect private sector expectations. Hence, the central bank faces a single-period problem: minimize

\[
\frac{1}{2} \left( \pi_t^2 + \lambda x_t^2 \right) \]  

subject to

\[
\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + e_t. \]  

(22)

The first-order conditions are taken, treating expectations as given. Those conditions imply

\[
\kappa \pi_t + \lambda x_t = 0. \]  

(23)

Note that this is the same condition as (6) for \( j = 0 \), the initial choice under fully optimal commitment (but not under the timeless perspective). The difference is that at \( t + 1 \), discretion implies

\[
\kappa \pi_{t+1} + \lambda x_{t+1} = 0, \]  

(24)

while precommitment implies

\[
\kappa \pi_{t+1} + \lambda (x_{t+1} - x_t) = 0. \]  

(25)

Use (23) in (2) to obtain

\[
x_t = \frac{\beta \lambda}{\lambda + \kappa^2} E_t x_{t+1} - \frac{\kappa}{\lambda + \kappa^2} e_t, \]  

(26)

where \( \left| \frac{\beta \lambda}{\lambda + \kappa^2} \right| < 1 \) since \( \frac{\beta \lambda}{\lambda + \kappa^2} = \frac{\beta}{1 + \kappa^2 \lambda} \) and \( 0 < \beta < 1 \). So the “forward solution” is stable and yields

\[
x_t = - \left( \frac{\kappa}{\lambda (1 - \beta \rho) + \kappa^2} \right) e_t \]  

(27)

for equilibrium output and
for equilibrium inflation and

\[ i_t = \left[ \frac{\sigma \kappa (1 - \rho) + \lambda \rho}{\lambda(1 - \beta \rho) + \kappa^2} \right] e_t + \sigma u_t \]  

(29)

for the equilibrium nominal interest rate.

Note that (28) implies the unconditional mean of inflation is zero, so there is no average inflation bias under discretion. But there is a stabilization bias, since the response to the cost shock, \( e_t \), is different under discretion that under precommitment.

Comparing the equilibrium interest rate under precommitment, (20), and (29):

- Under both precommitment and discretion, monetary policy completely offsets the impacts of the demand shock, \( u_t \), so \( u_t \) does not affect either \( x \) or \( \pi \).
- Cost shocks, \( e_t \), have different impacts under precommitment and discretion because under discretion there is a stabilization bias.
- There is history dependence of optimal policy under precommitment, but none under discretion. The history dependence is a means by which commitment is implemented.

3. An Extension

Finally, we consider an extension of the basic model in a couple of directions. First, we make interest rate variation costly in terms of the central bank’s objectives. Second, we introduce a control error in the policy instrument.\(^1\) Finally, we add an exogenous disturbance to the AS relation. That process is the sole source of persistence in the model.

For this extension, we consider only optimal policy under discretion.

We imagine that the central bank chooses \( i_t^* \), but that \( i_t \) is the realized interest rate, with

\[ i_t = i_t^* + \varepsilon_t. \]  

(30)

The realized interest rate, \( i_t \), affects private-sector decision, but \( i_t^* \) describes the central bank’s choice of interest rate, and (30) is added as an equation of the model. Now the central bank’s period loss function is

\(^1\)We do not mean to imply that central banks cannot perfectly control their policy instruments. This is an easy way to introduce a “policy shock” in a framework in which policy is optimal.
\[ L_t = \frac{1}{2} \left( \pi_t^2 + \lambda_x x_t^2 + \lambda_i (i_t^*)^2 \right). \]  
(31)

The central bank chooses \( \pi_t, x_t, i_t^* \) to minimize (31) subject to

\[ \pi_t = \kappa x_t + \beta E_t \pi_{t+1} + \alpha z_t + u_t \]  
(32)

\[ x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}). \]  
(33)

All parameters are positive, with the exception of \( \alpha \), which could be of any sign.

The first-order condition for optimal policy under discretion is

\[ i_t^* = \frac{\sigma^{-1} \lambda_x}{\lambda_i} x_t + \frac{\sigma^{-1} \kappa}{\lambda_i} \pi_t. \]  
(34)

This model can be solved using the method of undetermined coefficients. Further suppose that \( \varepsilon_t \), the control error for policy and \( u_t \) are i.i.d. random variables. The variable \( z \) obeys \( z_t = \rho z_{t-1} + \nu_t \), and provides the sole source of dynamics in the model. We posit decision rules of the form

\[ \pi_t = a_i \varepsilon_t + a_z z_t + a_u u_t, \]  
(35)

\[ x_t = b_i \varepsilon_t + b_z z_t + b_u u_t. \]  
(36)

These posited rules, together with the assumptions about the exogenous processes, imply that

\[ E_t \pi_{t+1} = a_z \rho z_t, \]  
(37)

\[ E_t x_{t+1} = b_z \rho z_t. \]  
(38)

Substituting (35)-(38) into (32) and (33) and solving for the unknown coefficients yields

\[ a_i = -\frac{\sigma^{-1} \kappa \lambda_i}{\lambda_i + \sigma^{-2} (\lambda_x + \kappa^2)}, \]  
(39)

\[ b_i = -\frac{\sigma^{-1} \lambda_i}{\lambda_i + \sigma^{-2} (\lambda_x + \kappa^2)}, \]  
(40)

\[ a_z = -\frac{\alpha \sigma^{-1} \kappa (\sigma^{-1} \kappa - \rho \lambda_i)}{(1 - \beta \rho)^2 [\lambda_i (1 - \rho) + \sigma^{-2} \lambda_x] + \kappa \sigma^{-1} (\kappa \sigma^{-1} - \rho \lambda_i)} + \frac{\alpha}{1 - \beta \rho}, \]  
(41)
\[ b_z = -\frac{\alpha \sigma^{-1}(\sigma^{-1} \kappa - \rho \lambda_i)}{(1 - \beta \rho)^2 \lambda_i(1 - \rho) + \sigma^{-2} \lambda_x} + \kappa \sigma^{-1}(\kappa \sigma^{-1} - \rho \lambda_i), \] (42)

\[ a_u = \frac{\lambda_i + \sigma^{-2} \lambda_x}{\lambda_i + \sigma^{-2}(\lambda_x + \kappa^2)}, \] (43)

\[ b_u = -\frac{\sigma^{-2} \kappa}{\lambda_i + \sigma^{-2}(\lambda_x + \kappa^2)}. \] (44)

These coefficients have the following sign patterns:

\[ a_i < 0, b_i < 0, \text{sign}(a_z) = \text{sign}(\alpha), \text{sign}(b_z) = \text{sign}(-\alpha), a_u > 0, b_u < 0. \] (45)

The “structural form” of the equilibrium can be written as

\[
\begin{bmatrix}
1 & 0 & 0 & -a_z \\
0 & 1 & 0 & -b_z \\
-\sigma^{-1} \kappa & -\sigma^{-1} \lambda_x & \lambda_i & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\pi_t \\
x_t \\
i_t \\
z_t
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\pi_{t-1} \\
x_{t-1} \\
i_{t-1} \\
z_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
a_i & a_u & 0 \\
b_i & b_u & 0 \\
\lambda_i & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon_t \\
\mu_t \\
\nu_t
\end{bmatrix}. \] (46)

Letting \( Y_t = (\pi_t, x_t, i_t, z_t)' \) and \( \zeta_t = (\varepsilon_t, u_t, \nu_t)' \), the reduced form for the equilibrium is

\[ Y_t = AY_{t-1} + B\zeta_t \]

\[ = AY_{t-1} + \eta_t \] (47)

or

\[
\begin{bmatrix}
\pi_t \\
x_t \\
i_t \\
z_t
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & \rho a_z \\
0 & 0 & 0 & \rho b_z \\
0 & 0 & 0 & \frac{\rho \sigma^{-1}}{\lambda_x} (\kappa a_z + \lambda_x b_z) \\
0 & 0 & 0 & \rho
\end{bmatrix}
\begin{bmatrix}
\pi_{t-1} \\
x_{t-1} \\
i_{t-1} \\
z_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
\eta_{\pi t} \\
\eta_{xt} \\
\eta_{it} \\
\eta_{zt}
\end{bmatrix}. \] (48)

The one-step-ahead forecast errors are given by
\[
\begin{bmatrix}
\eta_{\pi t} \\
\eta_{xt} \\
\eta_{it} \\
\eta_{zt}
\end{bmatrix} =
\begin{bmatrix}
\frac{a_i}{\lambda} (\kappa a_i + \lambda x b_i) + 1 \\
\frac{a_u}{\lambda} (\kappa a_u + \lambda x b_u) \\
\frac{a_z}{\lambda} (\kappa a_z + \lambda x b_z) \\
0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_t \\
u_t
\end{bmatrix} .
\]

Notice that an econometrician with data on \{\pi_t, x_t, i_t, z_t\} who estimated the reduced form (48) would conclude that interest rates fail to Granger-cause both inflation and output. From this the econometrician might conclude that monetary policy is irrelevant. This situation arises despite the fact that control errors in policy, the \varepsilon’s, have impacts on both \pi and \pi.

References
