A SIMPLE MODEL OF THE FISCAL THEORY OF THE PRICE LEVEL

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At the simplest level, the fiscal theory of the price level (FTPL) can be thought of as an application of the general point that Sargent and Wallace (1981) make: fiscal policy behavior may impose restrictions on what monetary policy can achieve. Conceived at this general level, it is difficult to understand why the FTPL has been so maligned. But this level of generality can also produce misleading understandings of the FTPL, as the mechanism and the details of the FTPL differ sharply from that emphasized by Sargent and Wallace. “Unpleasant Monetarist Arithmetic” is essentially the old story that when fiscal policy cannot (or will not) adjust to service government debt, then monetary policy must adjust by passively generating whatever seigniorage revenues are needed. The FTPL, though it may involve generating seigniorage revenues, is not primarily a story about the central bank monetizing fiscal deficits. For this reason, the often-voice argument that the FTPL does not apply to modern, developed countries because in those countries seigniorage is a trivial source of revenues (on the order of 1% of GDP according to King (1995)), is largely beside the point. The FTPL is consistent with any average level of inflation and seigniorage.

This note lays out the FTPL in the simplest model with monetary frictions and randomness in macro policies. The model is from Leeper (1991), but the note shows how to solve the model using Sims’s (2001) method for solving linear rational expectations models. In Leeper (1991) the model is solved in the usual way by iterating on stochastic difference equations.

1. The Model

A single, infinitely lived agent lives in an endowment version of a Sidrauski (1967) model. For simplicity, both the endowment, \( y \), and government consumption, \( g \), are constant. Assume \( g = 0 \). The agent chooses sequences \( \{c_t, M_t, B_t\} \) to solve:

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t \left[ \log(c_t) + \delta \log \left( \frac{M_t}{p_t} \right) \right], \quad 0 < \beta < 1, \delta > 0,
\]

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subject to
\[ c_t + \frac{M_t + B_t}{p_t} + \tau_t = y + \frac{M_{t-1} + R_{t-1}B_{t-1}}{p_t}. \] (2)

\( M \) is nominal money balances, \( B \) is nominal one-period government debt, which pays gross nominal interest at rate \( R \), \( c \) is consumption, and \( \tau \) is lump-sum taxes (if positive) and transfers (if negative). Initial liabilities, \( M_{-1} + R_{-1}B_{-1} > 0 \) are given.

The aggregate resource constraint for this economy, which is also the goods market clearing condition, is
\[ c_t + g_t = y. \] (3)

The first-order conditions imply the Fisher and money-demand equilibrium relations
\[ \frac{1}{R_t} = \beta E_t \left[ \frac{1}{\pi_{t+1}} \right], \] (4)
\[ m_t = \delta c \left[ \frac{R_t - 1}{R_t} \right]^{-1}, \] (5)
where \( \pi_t = \frac{p_t}{p_{t-1}} \) and \( m_t = \frac{M_t}{p_t} \).

Government policy is given by sequences \( \{M_t, B_t, \tau_t\} \), ignoring \( g \) since it is set to zero. These sequences must satisfy the government’s budget identity
\[ b_t + m_t + \tau_t = g + \frac{R_{t-1}b_{t-1} + m_{t-1}}{\pi_t}, \] (6)
where \( b_t = \frac{B_t}{p_t} \).

We describe government policies in terms of simple rules that determine their instruments. For fiscal policy we have
\[ \tau_t = \gamma_0 + \gamma b_{t-1} + \psi_t, \] (7)
where \( \psi_t \) is an exogenous shock, realized at the beginning of \( t \), that follows
\[ \psi_t = \rho \psi_{t-1} + \varepsilon_t, \] (8)
with \( |\rho_\psi| < 1 \) and \( E_t \varepsilon_{\psi t+1} = 0 \). Monetary policy is couched as obeying the interest rate rule
\[ R_t = \alpha_0 + \alpha \pi_t + \theta_t, \] (9)
where \( \theta_t \) is an exogenous shock, realized at the beginning of \( t \), that follows
\[ \theta_t = \rho_\theta \theta_{t-1} + \varepsilon_\theta, \] (10)
with $|\rho_\theta| < 1$ and $E_t\varepsilon_{\theta t+1} = 0$.\(^1\)

Equations (2)-(10) hold for $t \geq 0$. This requires that we specify initial values for the exogenous shocks: $\theta_{-1} = \psi_{-1} = 0$. Nothing rests on this. Analogously, the specification about implicitly assumes that some $p_{-1}$ is also given, as the model determines $\{\pi_t, b_t\}, t \geq 0$. It is natural to assume that $R_{-1}b_{-1} + m_{-1} > 0$ is given and arbitrary.

We shall show how the equilibrium depends on the policy parameters $(\alpha, \gamma)$.

2. Solving the Model

With policy rules (7) and (9), this non-linear model cannot be solved analytically. We use a first-order approximation to the model by linearizing around the deterministic steady state. This allows the model to be reduced to a dynamical system in $(\pi_t, b_t)$. Define the forecast error

$$\eta_{t+1} = \pi_{t+1} - E_t\pi_{t+1}. \quad (11)$$

After some judicious substitutions, the system to be solved may be written as

$$\begin{bmatrix} 1 & 0 \\ \varphi_1 & 1 \end{bmatrix} \begin{bmatrix} \pi_{t+1} \\ b_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha \beta & 0 \\ \varphi_2 & \beta^{-1} - \gamma \end{bmatrix} \begin{bmatrix} \pi_t \\ b_t \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ \varphi_3 & -1 \end{bmatrix} \begin{bmatrix} \theta_{t+1} \\ \psi_{t+1} \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ \varphi_4 & 0 \end{bmatrix} \begin{bmatrix} \theta_t \\ \psi_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \eta_{t+1}, \quad (12)$$

which holds for $t \geq 0$.

The $\varphi_i$ parameters in (12) are defined in terms of steady state variables as:

$$\varphi_1 = \frac{\delta y}{R-1} \left[ \frac{1}{\beta \pi} - \frac{\alpha}{R-1} \right] + \frac{\bar{b}}{\beta \pi}$$

$$\varphi_2 = -\frac{\alpha}{\bar{\pi}} \left[ \frac{\delta y}{(R-1)^2} - \bar{b} \right]$$

$$\varphi_3 = \frac{\delta y}{(R-1)^2}$$

$$\varphi_4 = \frac{\varphi_2}{\alpha} = -\frac{1}{\bar{\pi}} \left[ \frac{\delta y}{(R-1)^2} - \bar{b} \right]$$

\(^1\)Note that (9) is a simplified version of a Taylor (1993) rule.
To arrive at (12), in the second equation we have used expressions like (5), which hold only for \( t \geq 0 \), to substitute out real money balances. This is why we take \( R_{-1}b_{-1} + m_{-1} > 0 \) are given and arbitrary.

To use the notation in Sims (2001), let \( x_t = (\pi_t, b_t)' \) and \( z_t = (\theta_t, b_t)' \) and write (12) as

\[
\Gamma_0 x_{t+1} = \Gamma_1 x_t + \Phi_0 z_{t+1} + \Phi_1 z_t + \Pi \eta_{t+1},
\]

where the matrices have the obvious definitions.

Eigensystem analysis focuses on the transition matrix

\[
\Gamma_0^{-1} \Gamma_1 = \begin{bmatrix} \alpha \beta & 0 \\
- \alpha \beta \varphi_1 + \varphi_2 & \beta^{-1} - \gamma \end{bmatrix}.
\]

The lower triangular structure of (14) implies the eigenvalues can be read off the diagonal as being

\[
\alpha \beta \quad \text{and} \quad \beta^{-1} - \gamma.
\]

Because there is a single endogenous forecast error, \( \eta_{t+1} \), a unique equilibrium requires that one of these roots be larger than 1 and the other be smaller than one (in absolute value). There are four regions of the parameter space that are meaningful:

\begin{align*}
I : & \quad |\alpha \beta| \geq 1 \text{ and } |\beta^{-1} - \gamma| < 1 \\
II : & \quad |\alpha \beta| < 1 \text{ and } |\beta^{-1} - \gamma| \geq 1 \\
III : & \quad |\alpha \beta| < 1 \text{ and } |\beta^{-1} - \gamma| < 1 \\
IV : & \quad |\alpha \beta| \geq 1 \text{ and } |\beta^{-1} - \gamma| \geq 1
\end{align*}

Regions I and II deliver unique equilibria; Region III leaves the equilibrium indeterminate, allowing for sunspot equilibria; Region IV implies no equilibrium exists (unless the exogenous shocks, \( \varepsilon_{\psi t} \) and \( \varepsilon_{\theta t} \), are perfectly correlated in just the right way). In Region I, monetary disturbances produce the usual monetarist predictions and fiscal shocks are irrelevant; Ricardian equivalence holds. Region II describes the fiscal theory of the price level, where shocks to taxes generate inflation and monetary shocks generate rather non-monetarist impacts.
A straightforward way to solve the model using Sims’s algorithm is to stack the \( x \) and \( z \) vectors and embed the assumed processes for the exogenous disturbances in the model. Let \( Y_t = (\pi_t, b_t, \theta_t, \psi_t)' \) and \( \xi_t = (\eta_t, 0, \varepsilon_{\theta t}, \varepsilon_{\psi t})' \) and write

\[
\begin{bmatrix}
\pi_{t+1} \\
b_{t+1} \\
\theta_{t+1} \\
\psi_{t+1}
\end{bmatrix} = \begin{bmatrix}
\alpha \beta & 0 \\
-\alpha \beta \varphi_1 + \varphi_2 & \beta^{-1} - \gamma \\
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\beta & 0 \\
-\beta \varphi_1 + \varphi_4 & -\rho \\
0 & \rho \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\pi_t \\
b_t \\
\theta_t \\
\psi_t
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 & 0 \\
-\varphi_1 & 0 & \varphi_3 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\eta_{t+1} \\
0 \\
\varepsilon_{\theta t+1} \\
\varepsilon_{\psi t+1}
\end{bmatrix},
\] (16)

which holds for \( t = 0, 1, 2, \ldots \). More compactly, the system is

\[
Y_{t+1} = AY_t + C\xi_{t+1},
\] (17)

which implies

\[
Y_t = A^t Y_0 + \sum_{s=0}^{t-1} A^s C \xi_{t-s}.
\] (18)

The Jordan decomposition of \( A \) implies \( A^s = PA^s P^{-1} \), where the eigenvalues are along the diagonal of \( \Lambda \) (unless there are repeated roots, in which case the alignment is more complex). Let \( P^j \) be the \( j \)th row of \( P \) and let \( P'_j \) be the \( j \)th column of \( P \). Then (18) is

\[
Y_t = \sum_{j=1}^{n} P_j \lambda_j^t P'_j Y_0 + \sum_{j=1}^{n} P_j \sum_{s=0}^{t-1} \lambda_j^s P'_j C \xi_{t-s}.
\] (19)

To eliminate explosive eigenvalues (ones where \( |\lambda_j| > 1 \)) we need to impose for each explosive \( j \) :

\[
P^j Y_t = 0, \quad t = 0, 1, 2, \ldots
\] (20)

or, equivalently,

\[
P^j Y_0 = 0,
\] (21)

\[
P^j C \xi_t = 0, \quad t = 1, 2, \ldots
\] (22)
One way to understand this solution method is to note that we eliminated the conditional expectation in (4) by introducing a new variable, \( \eta_{t+1} \), the forecast error. To get a unique solution, we need a unique linear mapping from the \( \varepsilon \)'s to \( \eta \) (the exogenous structural errors to the endogenous reduced-form error). If the model supports more than one such mapping, the solution is not unique. If the model fails to generate a mapping (perhaps because it produces two or more mutually exclusive mappings), then no equilibrium exists.

The rules of thumb for existence and uniqueness are:

1. If there are \( q \) distinct (linearly independent) expectational errors—the \( \eta \)'s—then we need \( q \) unstable eigenvalues, which provide \( q \) additional restrictions of the form (21) and (22).
2. If there are fewer than \( q \) unstable roots, the model is underdetermined and the solution is not unique. There are too few additional restrictions to determine the \( q \) \( \eta \)'s.
3. If there are more than \( q \) unstable roots, the model is overdetermined and no solution exists. This is because too many additional restrictions are produced.

In this model, \( q = 1 \), so we need only one unstable root to uniquely determine the equilibrium.\(^2\)

One can show via simple algebra that

\[
P^{-1} = \begin{bmatrix}
1 & 0 & \frac{\beta}{\alpha \beta - \rho_\psi - (\beta^{-1} - \gamma)} \\
\frac{\alpha \beta \phi_3 - \phi_2}{\alpha \beta - (\beta^{-1} - \gamma)} & 1 & \frac{1}{\beta^{-1} - \gamma - \rho_\psi} \left[ \frac{\alpha \beta \phi_1 - \beta \phi_2}{\alpha \beta^{-1} - (\beta^{-1} - \gamma)} + \rho_\psi \phi_3 - \beta \phi_1 + \phi_4 \right] \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(23)

The first two rows of this matrix give us the stability conditions associate with Regions I and II, where unique equilibria exist.

3. **Equilibrium**

3.1. **Active Monetary Policy/Passive Fiscal Policy: Region I.** When \( |\alpha \beta| \geq 1 \) and \( |\beta^{-1} - \gamma| < 1 \), the first row of \( P^{-1} \) in (23) is the eigenvector associated with the unstable eigenvalue. The stability condition in this case is

\(^2\)It should be understood that because we are solving a version of the non-linear model that has been linearized around the steady state, all these results on existence and uniqueness are local results. For interesting work about global properties of models like the present one, see Benhabib, Schmitt-Grohe, and Uribe (2001a,b, 2002).
\[ P^1 Y_t = \begin{pmatrix} 1 & 0 & \frac{\beta}{\alpha \beta - \rho_\theta} & 0 \end{pmatrix} Y_t = 0, \quad t = 0, 1, 2, \ldots, \] (24)

implying that in equilibrium

\[ \pi_t = -\frac{\beta}{\alpha \beta - \rho_\theta} \theta_t \] (25)

and

\[ E_t \pi_{t+1} = -\frac{\beta \rho_\theta}{\alpha \beta - \rho_\theta} \theta_t, \] (26)

so that

\[ R_t = -\frac{\rho_\theta}{\alpha \beta - \rho_\theta} \theta_t. \] (27)

Surprise inflation is determined by the mapping from \( \varepsilon \) to \( \eta \):

\[ P^1 C \xi_t = \begin{pmatrix} 1 & 0 & \frac{\beta}{\alpha \beta - \rho_\theta} & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\varphi_1 & 0 & \varphi_3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_t \\ 0 \\ \varepsilon_{\theta t} \\ \varepsilon_{\psi t} \end{bmatrix} = 0, \quad t = 1, 2, \ldots, \] (28)

which implies

\[ \eta_t = -\frac{\beta}{\alpha \beta - \rho_\theta} \varepsilon_{\theta t}, \quad t = 1, 2, \ldots. \] (29)

This set of results makes clear that equilibrium inflation in Region I is entirely a monetary phenomenon. Tax disturbances do not affect inflation or nominal interest rates. Equilibrium sequences of \( \{\tau_t, b_t\} \) are determined by the (stable) difference equation in debt in (16) and the tax rule in (7). Because these sequences are irrelevant for inflation, the equilibrium exhibits Ricardian equivalence. A cut in taxes due to a negative realization of \( \psi_t \) raises \( b_t \), which in turn raises future lump-sum taxes. Note that when the policy shocks are i.i.d., equilibrium debt evolves according to

\[ b_t = \frac{1}{\alpha \beta \delta} \left( \frac{\delta y}{R - 1} + \bar{b} \right) \varepsilon_{\theta t} - \varepsilon_{\psi t} + \text{variables dated } t - 1. \] (30)

Higher \( \varepsilon_\theta \) raises \( R_t \), lowering \( m_t \) and raising \( b_t \)—this is an open-market sale—and through the tax rule, raises expected future taxes. Although taxes appear to be irrelevant, tax policy is far from irrelevant, as it supports monetary policy. The figure
below illustrates the dynamic impacts of i.i.d. monetary and fiscal policy shocks in Region I.

3.2. Passive Monetary Policy/Active Fiscal Policy: Region II. When $|\alpha \beta| < 1$ and $|\beta^{-1} - \gamma| \geq 1$, the second row of $P^{-1}$ in (23) is the eigenvector associated with the unstable eigenvalue. In the general case, this is a rather messy expression, so we will focus on the special case, which has also received the most attention in the fiscal theory literature. Suppose $\alpha = \gamma = \rho_\theta = \rho_\psi = 0$. With $\alpha = 0$, the nominal interest rate is exogenous, while with $\gamma = 0$, the net-of-interest fiscal surplus is exogenous. An exogenous surplus corresponds to the same assumption that Sargent and Wallace (1981) make about fiscal policy in their exposition of “Unpleasant Monetarist Arithmetic.” In this case, policy shocks today cannot generate changes in expected future taxes. This is an essential element of the fiscal theory.

The stability condition is

$$P^2 Y_t = \begin{pmatrix} 0 & 1 & -\frac{\delta y}{(R-1)^2} & 0 \end{pmatrix} Y_t = 0, \quad t = 0, 1, 2, ..., \quad (31)$$

implying that

$$b_t = \frac{\delta y}{(R-1)^2} \theta_t, \quad t = 0, 1, 2, .... \quad (32)$$

In equilibrium, shocks to taxes have no impact on the real value of government debt.

How does this happen? To understand this, first consider the mapping from $\varepsilon$ to $\eta$:

$$P^2 C \xi_t = \begin{pmatrix} 0 & 1 & -\frac{\delta y}{(R-1)^2} & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\varphi_1 & 0 & \varphi_3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_t \\ 0 \\ \varepsilon_{\theta t} \\ \varepsilon_{\psi t} \end{bmatrix} = 0, \quad t = 1, 2, ...., \quad (33)$$

so

$$\eta_t = \frac{1}{\varphi_1} \varepsilon_{\psi t}, \quad t = 1, 2, ...., \quad (34)$$

where $\varphi_1 > 0$, so a cut in taxes ($\varepsilon_{\psi t} < 0$) raises the forecast error and, therefore, current inflation. Taking expectations conditional on information at time $t$, the intertemporal government budget constraint is
\[
\frac{B_t}{p_t} = E_t \left\{ \frac{\pi_{t+1}}{R_t} \left[ \tau_{t+1} + s_{t+1} - g \right] + \sum_{s=1}^{\infty} \left( \prod_{k=1}^{s} \frac{\pi_{t+k+1} R_{t+k}^{-1}}{\tau_{t+s+1} - s_{t+s+1} - g} \right) \right\},
\]

where \( s \) is seigniorage revenues. Note that the first term on the right side involves \( \theta_t \) and future (unrealized) shocks, while the second term involves only future shocks. With \( \rho_\theta = \rho_\psi = 0 \), all future shocks are unanticipated, so conditional on information at \( t \), only \( \theta_t \) can affect the real value of government debt at \( t \). This is what (32) says.

The way that tax shocks at \( t \) leave the real value of debt unchanged is as follows. A surprise tax cut at \( t \) is financed by issuing new \textit{nominal} government debt. Monetary policy is pegging the interest rate, so \( R_t \) cannot change. This means there can be no change in expected inflation (and seigniorage). At the same time, fiscal policy is not allowing future taxes to change following the tax cut (that’s the meaning of \( \gamma = 0 \)). At the initial (pre-shock) prices and interest rates, the cut in taxes with no prospect of higher future taxes, leaves households feeling wealthier. Higher perceived wealth leads households to try to raise their consumption paths. In this endowment economy, the increase in demand for consumption goods can only result in higher goods prices. The price level rises until the change in wealth disappears, so in equilibrium, there is no change in real wealth and the complete impact of the tax cut is a rise in current inflation, as (34) reports.

Note that in this region of the parameter space the inflation process is stable and in the special case we are considering

\[
\pi_t = \beta \varepsilon_{\theta_{t-1}} - \frac{1}{\varphi_1} \varepsilon_{\psi t} \quad t = 0, 1, 2, \ldots
\]

A monetary policy shock that raises \( R_t \) (\( \varepsilon_{\theta_t} > 0 \)) and has only a delayed effect on inflation. In fact, the delayed effect is “perverse” by conventional monetary standards, as a higher interest rate at \( t \) raises inflation at \( t+1 \). Another way to see this is consider, using (35) the price-level raises of higher expected seigniorage \( s_{t+s+1} \). If no policies adjust at date \( t \), under the current assumptions on policy behavior, \( p_t \) must \textit{fall}. This is a very similar to the “unpleasant arithmetic” result that Sargent and Wallace derive.

The second figure below illustrates the impacts of \( i.i.d. \) policy disturbances in Region II.
Fixed Regime 1 (AM/PF): Policy Impacts
Fixed Regime 2 (PM/AF): Policy Impacts
REFERENCES


