Lecture 7

Sticky Price Models

1 A Basic Sticky Price Model without Capital

1.1 Households

The representative household chooses $C_{t+i}, C_{t+i}(z), N_{t+i}, M_{t+i}/P_{t+i}, B_{t+i}/P_{t+i}$ to max

$$E_t \left \{ \sum_{i=0}^{\infty} \beta^i \left[ \frac{1}{1-\gamma} C_{t+i}^{1-\gamma} + \frac{a_m}{1-\gamma_m} \left( \frac{M_{t+i}}{P_{t+i}} \right)^{1-\gamma_m} - \frac{a_n}{1+\gamma_n} N_{t+i}^{1+\gamma_n} \right] \right \}$$

subject to

$$C_t = \frac{W_t}{P_t} N_t + \Pi_t + TR_t - \frac{M_t - M_{t-1}}{P_t} - \frac{1}{1+i_t} B_t - B_{t-1},$$

where $\gamma, \gamma_m < 1$ and $\gamma_n > 0$.

Setting up the Bellman equation

$$v \left( \frac{M_{t-1}}{P_t}, \frac{B_{t-1}}{P_t} \right) = \max \left\{ \frac{1}{1-\gamma} C_t^{1-\gamma} + \frac{a_m}{1-\gamma_m} \left( \frac{M_{t+i}}{P_{t+i}} \right)^{1-\gamma_m} - \frac{a_n}{1+\gamma_n} N_{t+i}^{1+\gamma_n} + E_t \beta v \left( \frac{M_t}{P_{t+1}}, \frac{B_t}{P_{t+1}} \right) \right\}$$

subject to (2).

FOC

The first order conditions for the consumer’s problem are:

$N_t$:

$$\frac{W_t}{P_t} = \frac{a_n}{C_t^{1-\gamma}} N_t^{\gamma_n}$$

$B_{t+1}$:

$$C_t^{-\gamma} \frac{1}{1+i_t} = \beta v_2 \left( \frac{M_t}{P_{t+1}}, \frac{B_t}{P_{t+1}} \right) \frac{P_t}{P_{t+1}}$$

where the envelope condition is

$$v_2 \left( \frac{M_{t-1}}{P_t}, \frac{B_{t-1}}{P_t} \right) = C_t^{-\gamma}$$
updating the envelope condition one period forward and plugging it back into (5) we obtain

\[ 1 = E_t \left\{ (1 + i_t) \left( \frac{P_t}{P_{t+1}} \right) \beta \left( \frac{C_t}{C_{t+1}} \right)^{-\gamma} \right\} \tag{7} \]

or

\[ C_t^{-\gamma} = \frac{R_{t+1}}{P_{t+1}} \beta C_{t+1}^{-\gamma}, \tag{8} \]

where

\[ R_{t+1} = (1 + i_t) \frac{P_t}{P_{t+1}}. \tag{9} \]

\[ B_{t+1} : \]

\[ C_t^{-\gamma} = a_m \left( \frac{M_t}{P_t} \right)^{-\gamma_m} + \beta v_1 \left( \frac{M_t}{P_{t+1}} \frac{B_t}{P_{t+1}} \right) \frac{P_t}{P_{t+1}}. \tag{10} \]

where the envelope condition implies

\[ v_1 \left( \frac{M_{t-1}}{P_t} \frac{B_{t-1}}{P_t} \right) = C_t^{-\gamma}. \tag{11} \]

updating one period forward and plugging it into (10)

\[ a_m \left( \frac{M_t}{P_t} \right)^{-\gamma_m} = 1 - \frac{1}{1 + i_t} \tag{12} \]

Notice that if \( a_m = 0 \), for \( i_t > 0 \) the return from holding bonds dominates the return from holding real money balances. Equation (8) is a standard Euler equation. Equation (4) is an intratemporal condition capturing the consumption/leisure trade-off. It has the interpretation that the marginal rate of substitution between consumption and leisure be equal to the real wage. Equation (12) is the dynamic condition for the choice of money holdings. The marginal cost of foregoing one unit of consumption today must be equal to the pecuniary benefit of being able to buy an extra unit of consumption tomorrow, plus the nonpecuniary benefit measured by the current utility flow of an extra unit of money. Noting that from (8)

\[ \frac{P_t}{P_{t+1}} \beta C_{t+1}^{-\gamma} = \frac{C_t^{-\gamma}}{1 + i_t}, \]

we can write (12) as follows:

\[ C_t^{-\gamma} \left( \frac{i_t}{1 + i_t} \right) = a_m \left( \frac{M_t}{P_t} \right)^{-\varepsilon}. \tag{13} \]
1.2 Firms

1.2.1 Final Goods Producers

The economy is composed of a continuum of wholesale producers, whose total is normalized to unity. Final good producers ensemble the intermediate goods according to the following production function

\[ Y_f^t = \left[ \int_0^1 Y_f^t(z)^{\frac{\varepsilon}{\varepsilon-1}} \, dz \right]^{\frac{\varepsilon}{\varepsilon-1}} \]  

where \( \varepsilon > 1 \) is the price elasticity of demand. We can see how this is a CES production function, which also exhibits diminishing marginal product, property that will drive the firms to diversify and produce with all the intermediate goods available.

The final good producer will minimize its costs. Therefore it will choose \( Y_f^t(z) \) to

\[ \min \int_0^1 P_t(z) Y_f^t(z) \, dz \]

subject to

\[ \left[ \int_0^1 Y_t(z)^{\frac{\varepsilon}{\varepsilon-1}} \, dz \right]^{\frac{\varepsilon}{\varepsilon-1}} \geq \bar{Y} \]

Writing the langrangian

\[ L = \int_0^1 P_t(z) Y_f^t(z) \, dz - \lambda \left\{ \left[ \int_0^1 Y_t(z)^{\frac{\varepsilon}{\varepsilon-1}} \, dz \right]^{\frac{\varepsilon}{\varepsilon-1}} - \bar{Y} \right\} \]

FOC

The first order condition with respect to \( Y_f^t(z) \) is

\[ P_t(z) - \lambda \frac{\varepsilon}{\varepsilon-1} \left[ \int_0^1 Y_t(z)^{\frac{\varepsilon}{\varepsilon-1}} \, dz \right]^{\frac{\varepsilon}{\varepsilon-1}} - \frac{\varepsilon-1}{\varepsilon} Y_f^t(z)^{-\frac{1}{\varepsilon}} = 0 \]  

\[ P_t(z) = \lambda \frac{\left[ \int_0^1 Y_t(z)^{\frac{\varepsilon}{\varepsilon-1}} \, dz \right]^{\frac{\varepsilon}{\varepsilon-1}}}{\left[ \int_0^1 Y_t(z)^{\frac{\varepsilon}{\varepsilon-1}} \, dz \right]^{\frac{\varepsilon}{\varepsilon-1}}} Y_t^f(z)^{-\frac{1}{\varepsilon}} \]

\[ P_t(z) = \lambda \frac{Y_f^t(z)^{\frac{\varepsilon}{\varepsilon^2}}}{Y_t^f(z)} \]

3
\[ P_t(z) = \lambda \left( \frac{Y_t^f(z)}{Y_t^f} \right)^{-\frac{1}{\varepsilon}} \quad (16) \]

Now we need to solve for the lagrange multiplier. To do so, we write the first order condition in a different way

\[ P_t(z) = \lambda \frac{\partial Y_t^f}{\partial Y_t^f(z)} \quad (17) \]

\[ \frac{1}{Y_t^f} P_t(z) Y_t^f(z) = \lambda \frac{\partial Y_t^f}{\partial Y_t^f(z)} \frac{Y_t^f(z)}{Y_t^f} \]

\[ \frac{1}{Y_t^f} \int_0^1 P_t(z) Y_t^f(z) \, dz = \lambda \frac{1}{Y_t^f} \int_0^1 \frac{\partial Y_t^f}{\partial Y_t^f(z)} \frac{Y_t^f(z)}{Y_t^f} \, dz \]

\[ \frac{E_t}{Y_t^f} = \lambda \quad (18) \]

Since the final goods producers operate in perfect competition, total cost of production must be equal to the total value of the goods sold. Hence

\[ E_t = P_t Y_t^f \quad (19) \]

Combining (18) and (19) we obtain

\[ \lambda = P_t \quad (20) \]

Using this result into (16)

\[ P_t(z) = \left( \frac{Y_t^f(z)}{Y_t^f} \right)^{-\frac{1}{\varepsilon}} P_t \quad (21) \]

or

\[ Y_t^f(z) = \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t^f \quad (22) \]

**Market Demand of Intermediate Good z**

Integrating over all final good firms we can obtain the total demand of intermediate good z.

\[ Y_t(z) = \int_0^1 Y_t^f(z) \, df \]
\[ Y_t(z) = \int_0^1 \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_f^f \, df \]

\[ Y_t(z) = \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} \int_0^1 Y_f^f \, df \]

\[ Y_t(z) = \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \quad (23) \]

**Price Index**

If we plug the demand of good \( z \) by firm \( f \) into the production function of firm \( f \):

\[ Y_t^f = \left\{ \int_0^1 \left[ \frac{P_t(z)}{P_t} \right]^{-\varepsilon} Y_t^f \, dz \right\}^{\varepsilon-1} \]

\[ Y_t^f = \left( Y_t^f \right)^{\varepsilon-1} \int_0^1 \left( \frac{P_t(z)}{P_t} \right)^{-(\varepsilon-1)} \, dz \]

\[ Y_t^f = Y_t^f \left\{ \int_0^1 \left( \frac{P_t(z)}{P_t} \right)^{-(\varepsilon-1)} \, dz \right\}^{\varepsilon-1} \]

\[ 1 = \left[ \left( \frac{1}{P_t} \right)^{-(\varepsilon-1)} \int_0^1 [P_t(z)]^{-(\varepsilon-1)} \, dz \right]^{\varepsilon-1} \]

\[ 1 = \left( \frac{1}{P_t} \right)^{-\varepsilon} \left[ \int_0^1 [P_t(z)]^{-(\varepsilon-1)} \, dz \right]^{\varepsilon-1} \]

\[ P_t^\varepsilon = \left[ \int_0^1 [P_t(z)]^{1-\varepsilon} \, dz \right]^{\frac{1}{\varepsilon-1}} \]

\[ P_t = \left[ \int_0^1 [P_t(z)]^{1-\varepsilon} \, dz \right]^{\frac{1}{\varepsilon-1}} \quad (24) \]
1.2.2 Intermediate Good Producers

As we have seen, the intermediate good producer faces a downward sloping demand. This is due to the fact that the intermediate good market is monopolistically competitive.

The production function of an intermediate good firm \( z \) is

\[
Y_t(z) = A_t N_t(z) \quad (25)
\]

The market demand for firm \( z \) is

\[
Y_t(z) = \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \quad (26a)
\]

The firm will choose \( P_t(z), Y_t(z) \) and \( N_t(z) \) to

\[
\max E_{t-1} \left\{ \frac{P_t(z)}{P_t} Y_t(z) - \frac{W_t}{P_t} N_t(z) \right\}
\quad \text{st} \quad (25) \text{ and } (26a)
\]

If we plug (25) and (26a) into the optimization problem, and we using\(^1\) \( MC_t = \frac{w_t}{A_t} \), the problem of the firm becomes

choose \( P_t(z) \) to \( \max E_{t-1} \left\{ \frac{P_t(z)}{P_t} \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t - MC_t \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \right\} \quad (27) \)

FOC

The first order condition is

\[
E_{t-1} \left\{ (1 - \varepsilon) \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \frac{P_t}{P_t} + \varepsilon MC_t \left( \frac{P_t(z)}{P_t} \right)^{-1(1+\varepsilon)} Y_t \right\} = 0
\]

\[
E_{t-1} \left\{ \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \frac{P_t}{P_t} \left[ 1 - \varepsilon \right] + \varepsilon MC_t \left( \frac{P_t(z)}{P_t} \right)^{-1} \right\} = 0 \quad (28)
\]

if we devide both sides by \( (1 - \varepsilon) \) and define \( (1 + \mu) \equiv \frac{\varepsilon}{1 - \varepsilon} = \frac{1}{1+\varepsilon} \), we obtain

\[
E_{t-1} \left\{ \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \frac{P_t}{P_t} \left[ 1 + \varepsilon \frac{MC_t}{1 - \varepsilon} \left( \frac{P_t(z)}{P_t} \right)^{-1} \right] \right\} = 0
\]

or

\[
E_{t-1} \left\{ \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \frac{P_t}{P_t} \left[ 1 + (1 + \mu) MC_t \left( \frac{P_t(z)}{P_t} \right)^{-1} \right] \right\} = 0
\]

\(^1\)This condition comes from the firm minimizing cost.
where $\varepsilon = \frac{1}{1 - \varepsilon} = 1 + \mu$ is the mark-up of the firm, which is inversely related to the elasticity of demand, $\varepsilon$. In the case of perfect competition, when $\varepsilon = \infty$, $\frac{1}{1 - \frac{1}{\varepsilon}} = 1$, and hence $\mu$, the net mark-up over the marginal cost, is zero.

We can rewrite the FOC as

$$E_{t-1} \left\{ \left( \frac{P_t(z)}{P_t} \right)^{-\varepsilon} Y_t \left[ 1 + (1 + \mu) MC_t \left( \frac{P_t(z)}{P_t} \right)^{-1} \right] \right\} = 0$$

(29)

### 1.2.3 One period Sticky Prices

If firms set prices before the beginning of the period, there exists a difference between ex-ante and ex-post mark-up. In the absence of shocks or under flexible prices, the mark-up is constant and equal to the desired mark-up.

**Ex-Ante**

The condition that holds ex-ante and which determines the desired mark-up by the firm ($\mu$) is

$$E_{t-1} \left( \frac{P_t(z)}{P_t} \right) = E_{t-1} [(1 + \mu) MC_t]$$

(30)

**Ex-Post**

Ex-post, there exist a realization of $P_t$ and $W_t$, and we will have an ex-post mark-up

$$\frac{P_t(z)}{P_t}_{fixed \ at \ t} = (1 + \mu_t)_{ex- \ post \ mark-up} MC_t$$

(31)

Unanticipated increases in demand will drive up the marginal cost, through an increase in the demand of labor, and down the ex-post mark-up. Therefore we have counter-cyclical mark-up.

### 1.3 Government and Money Creation Process

**Government**

$$\frac{M_t - M_{t-1}}{P_t} = TR_t$$

(32)

**Money Creation Process**

$$\frac{M_{t+1}}{M_t} = 1 + g_m.$$
1.4 Symmetric Monopolistic Competitive Equilibrium

A symmetric equilibrium is characterized by the following conditions

\[ P_t(z) = P_t \quad \forall z, \quad (34) \]

\[ Y_t(z) = Y_t \quad \forall z, \quad (35) \]

\[ N_t(z) = N_t \quad \forall z, \quad (36) \]

\[ C_t = Y_t, \quad (37) \]

and

\[ B_t = 0. \quad (38) \]

Equation (38) is a bonds market clearing condition. In equilibrium the supply of bonds is zero. Everybody is indifferent between borrowing and lending. Since every individual is the same in this economy, aggregate saving is equal to zero in equilibrium. Equation (37) is a goods market clearing condition.

1.4.1 Aggregate Demand

The aggregate demand side of the economy is characterized by the following equations:

(1) Wide Economy Resource Constraint

\[ C_t = Y_t \quad (39) \]

(2) Euler Equation for Bonds

\[ C_t = E_t \left\{ (1 + i_t) \left( \frac{P_t}{P_{t+1}} \right) \beta (C_{t+1})^{-\frac{1}{\gamma}} \right\}^{-\sigma} \quad (40) \]

where \( \sigma = \frac{1}{\gamma} \).

1.4.2 Aggregate Supply

The aggregate supply side of the economy is characterized by the following equations:

(3) Aggregate Production Function

\[ Y_t = A_t N_t \quad (41) \]

(4) Labor Market Equilibrium

\[ A_t = (1 + \mu_t) a_n \frac{N_t^{\gamma_n}}{C_t^{-\gamma}} \quad (42) \]
which come from combining the ex-post FOC for the intermediate firm and the FOC for labor supply for the consumer:

\[
A_t = (1 + \mu_t) \frac{W_t}{P_t}
\]

\[
W_t \frac{P_t}{P_t} = a_n \frac{N_t^{\gamma_n}}{C_t^{-\gamma}}
\]

(5) Ex-ante Condition for Price Setting

\[
\hat{P}_t = (1 + \mu) E_{t-1} (MC_i^n)
\]

which comes from the ex-ante FOC for the intermediate firm, which expects \( \frac{P_t(z)}{P_t} = 1 \), and hence

\[
E_{t-1} \left( \frac{P_t(z)}{P_t} \right) = E_{t-1} [(1 + \mu) MC_t]
\]

\[
1 = E_{t-1} [(1 + \mu) MC_t]
\]

\[
\hat{P}_t = (1 + \mu) E_{t-1} (MC_i^n)
\]

Surprises in the nominal wages \( (W_t) \) or in technology \( (A_t) \) will cause the actual mark-up \( (\mu_t) \) to deviate from the desired mark-up \( (\mu) \).

(6) Euler Equation for Money

\[
\frac{\bar{M}_t}{\bar{P}_t} = C_t^{-\gamma_m} \left( 1 - \frac{1}{1 + i_t} \right)^{-\frac{1}{\gamma_m}} - \frac{1}{a_m} - \frac{1}{\gamma_m}
\]

where the nominal money stock \( (M_t) \) is fixed by policy and the level of price is fixed due to its sticky nature. We can see how, in this case, an anticipated shock in the money supply does not guarantee that the prices are going to move to adjust, since they are sticky. Hence, there exist an interaction between nominal money and real variables.

If we use the wide economy resource constraint (39) in both euler equations (40) and (44), we will obtain the IS and LM relations. Also combining the aggregate production function (41) and the labor market equilibrium (42) we obtain the Aggregate Supply.

IS

\[
Y_t = E_t \left\{ (1 + i_t) \left( \frac{P_t}{P_{t+1}} \right) \beta (Y_{t+1})^{-\frac{1}{\sigma}} \right\}^{-\sigma}
\]

LM

\[
\frac{\bar{M}_t}{\bar{P}_t} = Y_t^{-\gamma_m} \left( 1 - \frac{1}{1 + i_t} \right)^{-\frac{1}{\gamma_m}} - \frac{1}{a_m} - \frac{1}{\gamma_m}
\]

AS

\[
1 + \mu_t = \frac{1}{a_n} Y_t^{-(\gamma + \gamma_n)} A_t^{1 + \gamma_n}
\]

which shows how unanticipated increases in demand decrease the mark-up, pushing the economy temporarily closer to the competitive equilibrium.
1.5 Log-Linearization

The log-linearization of the model is done in the same manner as in Lecture 6, and therefore the derivations will be omitted.

**IS**

\[ \hat{y}_t = -\sigma \left[ \hat{i}_t - (E_t \hat{p}_{t+1} - \bar{p}_t) \right] + E_t \hat{y}_{t+1} \]  
(48)

where \( \bar{p}_t \) denotes the log-deviation of the price at period \( t \), which is fixed at \( t \) since prices are set at period \( t-1 \).

Since prices are fixed only for one period, the best forecast that economic agents can make about the future is the flexible price scenario. If we denote \( \hat{y}_t^* \) as the log-deviation of output from the steady state for the flexible price model, the best forecast will be

\[ E_t \hat{y}_{t+1} = E_t \hat{y}_{t+1}^* \]

Using this last equation and the Fisher parity condition

\[ \hat{i}_t = \hat{r}_t + E_t \hat{p}_{t+1} - \bar{p}_t \]  
(49)

we can re-write the IS as follows

\[ \hat{y}_t = -\sigma \hat{i}_t + E_t \hat{y}_{t+1}^* \]  
(50)

**LM**

\[ \hat{m}_t - \bar{p}_t = a \hat{y}_t - \alpha \hat{i}_t \]  
(51)

where \( a = \gamma_m \) and \( \alpha = \frac{1}{\gamma_m} \).

**AS**

\[ \hat{\mu}_t = - (\gamma + \gamma_n) \hat{y}_t + (1 + \gamma_n) \hat{a}_t \]  
(52)

In the flexible price equilibrium the ex-post mark-up does not deviate from the desired mark-up, and therefore the AS becomes

\[ 0 = - (\gamma + \gamma_n) \hat{y}_t^* + (1 + \gamma_n) \hat{a}_t \]  
(53)

where \( \hat{y}_t^* \) is the log-deviation of output from the steady state in the case of flexible prices. If you we plug (53) into (52) we obtain

\[ \hat{\mu}_t = - (\gamma + \gamma_n) (\hat{y}_t - \hat{y}_t^*) \]  
(54)

We know, by equation (31), that the ex-post mark-up is equal to the inverse of the marginal cost. Hence, combining the log-linearized version of (31)\(^2\) and (54)

\[ \hat{m}_c_t = (\gamma + \gamma_n) (\hat{y}_t - \hat{y}_t^*) \]  
(55)

**Money Growth**

\[ \hat{m}_t - \hat{m}_{t-1} = g_m \]  
(56)
1.5.1 Absence of Technology Shocks

In the absence of technology shocks, \( \hat{a}_t = 0 \) for all \( t \), we obtain

\[
E_t \hat{y}_{t+1} = \left( \frac{1 + \gamma_n}{\gamma + \gamma_n} \right) E_t \hat{a}_{t+1} = 0
\]

and the IS becomes

\[
\hat{y}_t = -\sigma \hat{r}_t = -\sigma [\hat{r}_t - (E_t \hat{p}_{t+1} - \bar{p}_t)]
\] (57)

If there is only unexpected shocks in the economy, agents will create expectations considering that the future is like the flexible price model. Hence, updating the LM one period forward

\[
\hat{m}_{t+1} - \hat{p}_{t+1} = a \hat{y}_{t+1} - \alpha \hat{i}_{t+1}
\]

where

\[
\hat{i}_{t+1} = \hat{i}_{t+1}^* + E_{t+1} \hat{p}_{t+2}^* - \hat{p}_{t+1}^*
\]

but, if \( \hat{a}_t = 0 \) for all \( t \), then \( \hat{y}_{t+1} = 0 \), and updating (57) one period forward, \( \hat{r}_{t+1}^* = 0 \). Hence

\[
\hat{m}_{t+1} - \hat{p}_{t+1} = -\alpha \hat{i}_{t+1}^*
\]

and

\[
\hat{r}_{t+1}^* = E_{t+1} \hat{p}_{t+2}^* - \hat{p}_{t+1}^*
\]

therefore

\[
\hat{p}_{t+1}^* = \hat{m}_{t+1} + \alpha \left( E_{t+1} \hat{p}_{t+2}^* - \hat{p}_{t+1}^* \right)
\] (58)

If money growth is constant

\[
E_{t+1} \hat{p}_{t+2}^* - \hat{p}_{t+1} = \hat{p}_{t+1}^* - \hat{p}_t = \hat{m}_{t+1} - \hat{m}_t = g_m
\] (59)

using (59) into (58) we obtain

\[
\hat{p}_{t+1}^* = \hat{m}_{t+1} + \alpha g_m = \hat{m}_t + (1 + \alpha) g_m
\]

Finally, since the best forecast of future prices is the flexible price case, \( E_t \hat{p}_{t+1} = \hat{p}_{t+1}^* \), equation (57) becomes

\[
\hat{y}_t = -\sigma [\hat{r}_t - \hat{m}_t - (1 + \alpha) g_m + \bar{p}_t]
\] (60)

Consider an increase in \( \hat{m}_t \):

1. the standard IS-LM analysis applies;

2. as \( \hat{m}_t \) increases, agents form expectations about future inflation. As a result, the real interest rate decreases to make money more attractive for given real money balances;

3. beliefs about future demand matter as well (through \( y_{t+1} \)). Notice that this effect will be clearer when prices will be fixed for more than one period.

From (60) when \( \hat{m}_t \) increases, \( y_t \) increases as well. With sticky prices, money affects real output. Furthermore, given \( \hat{m}_t \), if \( \mu \) increases, expected inflation increases as well. This puts a downward pressure on the real interest rate, so that real output increases.


1.6 Staggered Long-Term Pricing

Following the Calvo [1] setup, firms adjust their prices infrequently. The opportunity to adjust follows a Bernoulli distribution. Define $\theta$ the probability of keeping prices constant and $(1 - \theta)$ the probability of changing prices. In other words, each period there is a constant probability $(1 - \theta)$ that the firm will be able to adjust its price, independently of past history. This implies that the fraction of retailers setting prices at $t$ is $(1 - \theta)$. Thus, only a fraction of firms is setting prices at a certain period of time. The draw is independent of history and we do not need to keep track of firms changing prices. The time that elapses between price adjustments follows a geometric distribution. The expected time over which the price is fixed, i.e., the expected waiting time for the next price adjustment is therefore $\frac{1}{1 - \theta}$. The problem of the firm changing price at time $t$ consists of choosing $P_t(z)$ to max

$$E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t+i} \left[ \frac{P_t(z)}{P_{t+i}} Y_{t,t+i}(z) - \frac{W_{t+i}}{P_{t+i}} Y_{t,t+i}(z) \right]$$

subject to

$$Y_{t,t+i}(z) = \left[ \frac{P_t(z)}{P_{t+i}} \right]^{-\varepsilon} Y_{t+i},$$

where

$$\Lambda_{t,t+i} = \left( \frac{C_{t+i}}{C_t} \right)^{-\gamma}.$$  

In (62), $Y_{t,t+i}(z)$ is the firm’s demand function for its output at time $t+i$, conditional on the price set $i$ periods before at time $t$, i.e., $P_t(z)$. $\beta \Lambda_{t,t+i}$ is the relevant discount factor between $t$ and $t+i$. Using the cost-minimization condition $MC_{t+i} = \frac{W_{t+i}}{\Lambda_{t+i}}$, and substituting (62) into (61), the objective function can be written as:

$$E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,i} \left\{ \left[ \frac{P_t(z)}{P_{t+i}} \right]^{1-\varepsilon} Y_{t+i} - \frac{MC_{t+i}^n}{P_{t+i}} \left[ \frac{P_t(z)}{P_{t+i}} \right]^{-\varepsilon} Y_{t+i} \right\}.$$  

The FOCs write as

$$E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,i} Y_{t+i} \left\{ 1 - \varepsilon \frac{P_t(z)}{P_{t+i}} \left[ \frac{P_t(z)}{P_{t+i}} \right]^{-\varepsilon} Y_{t+i} + \varepsilon \frac{MC_{t+i}^n}{P_{t+i}} \left[ \frac{P_t(z)}{P_{t+i}} \right]^{-\varepsilon-1} \right\} = 0.$$  

Multiplying (65) by $P_t(z)$, dividing by $(1 - \varepsilon)$, and then simplifying:

$$E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,i} Y_{t+i} \left\{ \frac{P_t(z)}{P_{t+i}} \left[ \frac{P_t(z)}{P_{t+i}} \right]^{-\varepsilon} - \frac{\varepsilon}{\varepsilon - 1} \frac{MC_{t+i}^n}{P_{t+i}} \left[ \frac{P_t(z)}{P_{t+i}} \right]^{-\varepsilon} \right\} = 0.$$  

Then

$$E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,i} \left[ \frac{P_t(z)}{P_{t+i}} \right]^{1-\varepsilon} Y_{t+i} = (1 + \mu) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,i} \frac{MC_{t+i}^n}{P_{t+i}} \left[ \frac{P_t(z)}{P_{t+i}} \right]^{-\varepsilon} Y_{t+i},$$  

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where
\[ 1 + \mu = \frac{\varepsilon}{\varepsilon - 1}. \] (68)

Now, multiply and divide the RHS of (67) by \( \frac{P_t(z)}{P_{t+i}} \):
\[ E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left[ \frac{P^*_t(z)}{P_{t+i}} \right]^{1-\varepsilon} Y_{t+i} = (1 + \mu) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \frac{1}{P^*_t(z)} MC^m_{t+i} \left[ \frac{P^*_t(z)}{P_{t+i}} \right]^{1-\varepsilon} Y_{t+i}. \] (69)

Since \( P_t(z) \) does not depend on \( i \), we can take it out of the summation and bring it to the LHS of (69):
\[ P^*_t(z) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left[ \frac{P^*_t(z)}{P_{t+i}} \right]^{1-\varepsilon} Y_{t+i} = (1 + \mu) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} MC^m_{t+i} \left[ \frac{P^*_t(z)}{P_{t+i}} \right]^{1-\varepsilon} Y_{t+i}. \] (70)

We can simplify the terms \( P_t(z)^{1-\varepsilon} \) which appears in both sides and write
\[ P^*_t(z) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i} = (1 + \mu) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}.MC^m_{t+i}. \] (71)

Thus, the FOC can be written as
\[ P^*_t(z) = (1 + \mu) \frac{E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}.MC^m_{t+i}}{E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}}. \] (72)

or also
\[ P^*_t(z) = (1 + \mu) E_t \sum_{i=0}^{\infty} \omega_{t,t+i}.MC^m_{t+i}. \] (73)

The optimal price \( P_t(z) \) is a markup over a weighted average of expected future nominal marginal costs (nominal wages), where the markup is \( (1 + \mu) = \frac{\varepsilon}{\varepsilon - 1} \) (which corresponds to the desired markup under flexible prices) and where the weight \( \omega_{t,t+i} \) is
\[ \omega_{t,t+i} = \frac{(\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}}{E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}}. \] (74)

Note that:

1. for \( \theta = 0 \) the pricing equation becomes
\[ P_t = \frac{\varepsilon}{\varepsilon - 1} W_t \] (75)
as in the flexible price case;
2. the optimal price depends on forecasted future values of aggregate variables \((Y_t, P_t)\) as well as future marginal costs.

Equivalently, one can express the prices set by forward-looking firms relative to the overall price index \(P_t\). One can divide both sides of (71) by \(P_t\):

\[
\frac{P^*_t(z)}{P_t} E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i} = (1 + \mu) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}.MC_{t+i}^n. \tag{76}
\]

We could then multiply and divide each term of the RHS of (76) by \(P_{t+i}\) for \(i = 1, 2, \ldots\):

\[
\frac{P^*_t(z)}{P_t} E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i} = (1 + \mu) E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}.MC_{t+i+1} \frac{P_{t+i}}{P_t}. \tag{77}
\]

We can note that \(\frac{P_{t+i}}{P_t}\) is the cumulative gross inflation rate between \(t\) and \(t+i\). We can write:

\[
\frac{P_{t+i}}{P_t} = \prod_{j=0}^{i-1} \frac{P_{t+j+1}}{P_{t+j}} = \prod_{j=1}^{i} \Pi_{t+j} = \Pi_{t,t+i}. \tag{78}
\]

We can then write the FOC as

\[
\frac{P^*_t(z)}{P_t} = (1 + \mu) \frac{E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i},.MC_{t+i+1}}{E_t \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,t+i} \left( \frac{1}{P_{t+i}} \right)^{1-\varepsilon} Y_{t+i}}. \tag{79}
\]

or also

\[
\frac{P^*_t(z)}{P_t} = (1 + \mu) E_t \sum_{i=0}^{\infty} \omega_{t,t+i},.MC_{t+i+1}. \tag{80}
\]

The optimal relative price \(\frac{P_t(z)}{P_t}\) is a markup over a weighted average of expected future real marginal costs (real wages), and the weights are now \(\omega_{t,t+i},.\Pi_{t,t+i}\).

In equilibrium each producer that chooses a new price \(P_t(z)\) in period \(t\) will choose the same new price \(P_t(z)\) and the same level of output. Then the dynamics of the consumption-based price index will obey

\[
P_t = \left[ \theta P_{t-1}^{1-\varepsilon} + (1 - \theta) P^*_t(z)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}. \tag{81}
\]

**Log-linearization**

Let’s log-linearized the FOC of the firm. From equation (71) we can write

\[
\frac{P^*_t(z)}{P_t} E_t \left\{ \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,i} Y_{t+i} \frac{P^*_t(z)^{1-\varepsilon}}{P_{t+i}} \right\} = (1 + \mu) E_t \left\{ \sum_{i=0}^{\infty} (\theta \beta)^i \Lambda_{t,i} \frac{MC_{t+i}^n}{P_t} Y_{t+i} \frac{P^*_t(z)^{1-\varepsilon}}{P_{t+i}} \right\}. \tag{82}
\]
Consider first the left-hand side (LHS) of the above equation, and note that we drop any product of two or more variables, since we are using a first order approximation:

\[
\text{LHS} \simeq \left( P^eY \right) \left[ \sum_{i=0}^{\infty} (\theta \beta)^i \right] (\hat{p}_t^* - \hat{p}_t) + \left( P^eY \right) \sum_{i=0}^{\infty} (\theta \beta)^i \left[ \hat{y}_{t+i} + (\varepsilon - 1)\hat{p}_{t+i} + \hat{\lambda}_{t,i} \right].
\]

Consider now the right-hand side (RHS). Applying the same logic\(^3\):

\[
\text{RHS} \simeq \left( P^eY \right) \sum_{i=0}^{\infty} (\theta \beta)^i \left[ (\hat{m}c_{t+i}^n - \hat{p}_t) + \hat{y}_{t+i} + (\varepsilon - 1)\hat{p}_{t+i} + \hat{\lambda}_{t,i} \right].
\]

Combining LHS and RHS, simplifying, and dividing through by \( P^eY \) we get

\[
\hat{p}_t^* = (1 - \theta \beta)E_t \left\{ \sum_{i=0}^{\infty} (\theta \beta)^i \left( \hat{m}c_{t+i}^n \right) \right\},
\]

which gives the log-deviation of the newly set price as a discounted stream of the log-deviation of the nominal marginal cost.

Log-linearizing the price index (81) yields

\[
\hat{p}_t = \theta \hat{p}_{t-1} + (1 - \theta)\hat{p}_t^*
\]

Combine these last two equations to obtain:

\[
\hat{\pi}_t = \delta \hat{m}c_t + \beta E_t \hat{\pi}_t+1,
\]

with

\[
\delta = \frac{(1 - \theta)(1 - \beta \theta)}{\theta}.
\]

and where \( \hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1} \).

Using equation (55), we can write the log-linearized aggregate supply (82) as

\[
\hat{\pi}_t = k \left( \hat{y}_t - \hat{y}_t^* \right) + \beta E_t \hat{\pi}_{t+1}
\]

This equation represents the AS curve. It can be observed that:

1. as output increases, inflation goes up;
2. it is a forward looking AS since

\[
\hat{\pi}_t = k E_t \sum_{i=0}^{\infty} \beta^i \left( \hat{y}_{t+i} - \hat{y}_{t+i}^* \right)
\]

Inflation depends on future beliefs about capacity, meaning that inflation is a forward looking variable;

\(^3\)Note that we switch to real marginal cost by dividing and multiplying by \( P_{t+i} \).
3. there is no trade-off problem for the Central Bank. To control inflation the central bank does not need to generate a recession. Stabilizing output the central bank is also stabilizing inflation.

The complete log-linear system of the dynamic sticky price model is: the IS curve:

\[ \dot{y}_t = -\sigma \dot{r}_{t+1} + E_t \dot{y}_{t+1} \quad (84

the LM curve:

\[ \dot{m}_t - \dot{p}_t = a \dot{y}_t - \alpha \dot{\alpha}_t \quad (85) \]

the Fisher parity condition:

\[ \dot{r}_t = \dot{r}_{t+1} + E_t \dot{p}_{t+1} - \dot{p}_t \quad (86) \]

the forward-looking Phillips curve:

\[ \dot{\pi}_t = k \dot{y}_t + \beta E_t \dot{\pi}_{t+1} \quad (87) \]

the inflation rate:

\[ E_t \dot{\pi}_{t+1} = E_t \dot{p}_{t+1} - \dot{p}_t \quad (88) \]

exogenous process for money growth:

\[ \dot{m}_{t+1} - \dot{m}_t = g_m. \quad (89) \]

References


