Topics

(1) instantaneous rate of change
(2) the derivative function
(3) What the derivative of a function tells us about a function.
(4) Newton and Leibniz notation for derivatives
(5) local linear approximation
(6) relative rate of change

• Section 2.1, page 95: 1, 3, 5, 6, 8, 9, 11–13, 19, 20, 23
• Section 2.2, pages 101: 1, 3, 5, 7, 9, 10, 13, 15, 17–22, 25, 26
• Section 2.3, page 108: 3, 6, 7, 17, 19, 23, 28, 29, 43, 47, 49

Warning: We have dealt with two types of rate in this course, absolute rates of change such as velocity, and relative rates of change such as interest rates. In the case of population growth, we might discuss both the absolute rate of change, measured in people per unit time, and the relative rate of change, measured in percent per unit time. Linear functions are characterized by constant absolute rate of change. Exponential functions are characterized by constant relative rate of change. In this section we will focus on absolute rates of change, but at the end we will connect this with relative rates of change.

Review Examples 19 and 20 from the Week 1 notes about the traveller on Interstate 70. We computed

\[
\text{average speed} = \frac{\text{distance travelled}}{\text{elapsed time}}.
\]

We also graphed distance from Terre Haute as a function of time.

• For a trip made at constant speed, the graph is a straight line. The slope of this line is the speed.
• For a trip made at varying speed, the graph is a curve. The steepness of the curve at a point corresponds to the speed at a particular time.

Question 1: For the first distance vs. time graph in Example 20, was average speed higher during the first hour, or the last hour?
Question 2: Imagine a series of electronic toll collection stations along the highway. How might we use the toll stations to determine whether a motorist has been speeding?

Question 3: Can you devise a method based on these ideas for measuring the speed of the motorist at a single point in time, without using a speedometer or radar gun?

Displacement and velocity
There are two points on Interstate 70 that are a distance of 10 miles from Terre Haute, one to the east and one to the west. To distinguish these points from each other, we adopt the convention that positive numbers are used for points in one direction, say east, and negative numbers for points in the other direction, say west. Imagine Route 70 as a number line, with the origin representing Terre Haute.

We call a measurement of position in which the sign of the measurement is used to indicate direction as a **displacement** rather than a distance.

We may also use signs to indicate the direction of motion. A measurement of how fast an object is moving that uses sign to indicate the direction of motion is called a **velocity** rather than a speed. In our example, a car moving eastward would have positive velocity and a car moving westward would have negative velocity. We have

\[
\text{average velocity} = \frac{\text{net displacement}}{\text{elapsed time}}.
\]

**Example 1:** A sightseer sets out from her hotel at noon, driving along a scenic road, eventually reaching a destination 100 miles distant at 3:00 PM. She then turns around, driving along the same road, finally reaching her hotel at 5:00 PM.

What was the distance travelled between noon and 3:00 PM and what was the displacement, taking the hotel as reference point and the direction toward the destination as the positive direction?

What was the average speed between noon and 3:00 PM? The average velocity?
What was the total distance travelled between noon and 5:00 PM? The total displacement?

What was the average speed between noon and 5:00 PM? The average velocity?

**Instantaneous velocity**

We now address Question 3: How do we compute the velocity at a particular moment in time if we know displacement as a function of time?

**Example 2:** Consider the motion of a rock dropped from a tall bridge. Experience tells us that the rock picks up speed as it falls. To a good approximation, the displacement $y$ of the rock relative to the position at which it was released is

$$y(t) = 5t^2.$$ 

Here $y$ is measured in meters; we take $y = 0$ to be the bridge level and the downward direction to be positive; $t$ is time since release, measured in seconds.

This formula allows us to calculate the position of the rock at various times.

Now let us see whether we can figure out the velocity of the rock at a particular instant in time, say $t = 2$ seconds. The displacement of the rock at this time is $y(2) = 5 \cdot 2^2 = 20$ meters.

We can get a rough idea of the velocity at time $t = 2$ by determining how far the rock moves during the tenth of a second between $t = 2$ and $t = 2.1$. At time $t = 2.1$ the rock is $5 \cdot 2.1^2 = 22.05$ meters below the bridge, and so has moved 2.05 meters in that tenth of a second. Its average velocity during that interval is therefore

$$\frac{\Delta y}{\Delta t} = \frac{y(2.1) - y(2)}{2.1 - 2} = \frac{22.05 - 20}{2.1 - 2} = \frac{2.05}{.1} = 20.5 \text{ meters/sec}.$$ 

We know that this is only an approximation since the rock picks up speed during the interval. In fact, it will be an overestimate.

An obvious way to improve the accuracy of the approximation is to compute the average velocity over a shorter interval, say 1/100 of a second. The rock will pick up less speed in a shorter interval. Hence the average over this interval will be closer to the true velocity at time $t = 2$. Let’s do this calculation. We need the position at time $t = 2.01$ which is $y(2.01) = 5 \cdot 2.01^2 = 20.2005$.

$$\frac{\Delta y}{\Delta t} = \frac{y(2.01) - y(2)}{2.01 - 2} = \frac{20.2005 - 20}{2.01 - 2} = \frac{.2005}{.01} = 20.05 \text{ meters/sec}.$$
Since we have the formula for \( y(t) \) there is no limit to the accuracy we can obtain. All we need to do is to make the time interval shorter and shorter. Here is a table of the velocity estimate for various time intervals.

<table>
<thead>
<tr>
<th>( \Delta t )</th>
<th>.1</th>
<th>.01</th>
<th>.001</th>
<th>.0001</th>
<th>.00001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta y/\Delta t )</td>
<td>20.5</td>
<td>20.05</td>
<td>20.005</td>
<td>20.0005</td>
<td>20.00005</td>
</tr>
</tbody>
</table>

We have got pretty far with the realization that to find the velocity at an instant in time we need to consider how much movement occurs between that instant and another instant a short time later. This only gets us an approximation, however, whereas we really would like an exact answer. It’s pretty clear from the above data what the exact answer should be: the velocity estimate is getting closer to the value 20 each time. This is where the notion of **limit** enters:

If in making an approximation better and better, the approximate value gets arbitrarily close to a certain number, \( \ell \), then \( \ell \) is called the limit of the approximation.

It can be proved (we will do so later) that the limit of the above approximation is indeed 20. We have therefore determined that the **instantaneous velocity** at \( t = 2 \) is 20. Instantaneous velocity is an example of the more general concept of **instantaneous rate of change** of one quantity with respect to another. The instantaneous rate of change in \( y \) at time 2 is denoted \( y'(2) \). The symbol ‘ is called “prime”.

In mathematical notation, the procedure we carried out would be expressed as

\[
y'(2) = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = 20.
\]

**Example 3:** Repeat the procedure for \( t = 1 \) and \( t = 3 \). Can you guess a formula for \( y'(t) \)?

Both the displacement and the velocity are functions of time. The displacement is \( y(t) \) and the velocity is \( y'(t) \). The function \( y'(t) \) is called the **derivative** of the function \( y(t) \). A function and its derivative function are, of course, intimately tied together, and their relationship is a major topic of this course.

**Estimating the derivative from a tabulated function**

In Example 2, we had the advantage of having an exact formula for the quantity whose derivative we were trying to find. If instead we only have tabulated values, we will not be able to perform the limiting process. We will have to make do with the values given in the table.
Example 4: Recall from the first lecture the table of the population of Monroe County as a function of time

<table>
<thead>
<tr>
<th>$t$</th>
<th>1900</th>
<th>1910</th>
<th>1920</th>
<th>1930</th>
<th>1940</th>
<th>1950</th>
<th>1960</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(t)$</td>
<td>20,873</td>
<td>23,426</td>
<td>24,519</td>
<td>35,974</td>
<td>36,534</td>
<td>50,080</td>
<td>59,225</td>
</tr>
</tbody>
</table>

Suppose we wanted to find the number of people per year by which the population was growing in 1930. We cannot carry out the limiting process because the minimum spacing between data points is 10 years. The best we can do is to approximate the derivative $P'(1930)$ by computing the average rate of change on the shortest available interval.

We actually have two choices of interval: $[1920,1930]$ or $[1930,1940]$. The former is called the left estimate and the latter the right estimate. Generally, the best accuracy is obtained by averaging the left and right estimates. An equivalent procedure would be to do a two-sided estimate using the interval $[1920,1940]$.

Warning: The two-sided estimate is the same as the average of left and right estimates only when the $\Delta t$ used in the left estimate is the same as the $\Delta t$ used in the right estimate. This is the case in our example since $\Delta t = 10$ in both cases.

Compute the left estimate, the right estimate, the average of the two, and the two-sided estimate. Verify that the last two are equal.

The geometric meaning of the derivative

Recall that the average rate of change has geometric meaning: it is the slope of the secant line connecting the two points on the graph of a function corresponding to the endpoints of the interval over which the average is taken. If one of the endpoints is moved, the slope of the secant line will change. As the interval is made shorter and shorter by moving one of the endpoints towards the other, the slope some limiting value of the slope. Let’s try to characterize the line with the limiting slope geometrically.

Example 5: Estimate the the slope at $t = 3$ of the function $y(t) = 4 + 3t - t^2$ using smaller and smaller intervals with $(3, 4)$ as one of the endpoints. (Refer to computer demonstration.)

The derivative of a function at a point is the slope of the line tangent to the curve at that point.
Example 6: Using the interpretation of the derivative as the slope of the tangent line, would you say that the derivative of the function in Example 5 is positive, zero, or negative at the points 0, 1, 1.5, 2, 3?

So far we have learned about computing derivatives by using the average rate of change formula and then decreasing the size of the interval used until we could determine the limiting value of the rate. In our example the function whose derivative we computed was the distance fallen by a rock dropped from a bridge, \( y(t) = 5t^2 \). We computed the derivative at several points in time. By spotting a pattern in the results, we guessed a formula for the derivative at any time, \( t \). (Our guess was \( y'(t) = 10t \).) This is called the derivative function.

The derivative has many aspects:

- The derivative of a function \( f(x) \) at a particular \( x \) value, say \( x = 3 \), is a number. It describes how fast \( f(x) \) is changing at \( x = 3 \). If \( x \) represents time and \( f(x) \) represents the displacement of an object, then the derivative \( f'(3) \) represents the velocity of the object at time \( x = 3 \).
- The derivative function, \( f'(x) \) describes how the instantaneous rate of change of \( f(x) \) depends on the independent variable \( x \). The derivative function of the displacement function is the velocity function.
  - When \( f(x) \) is given by a formula, it is possible to find a formula for \( f'(x) \) by applying certain rules to the formula for \( f'(x) \). We will learn these rules in Chapter 3.
  - When \( f(x) \) is given by a graph, it is possible to draw the graph of \( f'(x) \). We do this by graphing the changing slope of the \( f(x) \) graph as a function of \( x \). We will practice this today.
  - When \( f(x) \) is given by a table, the best we can do is to approximate \( f'(x) \). By recording these estimates in a second table, we can get some idea of the shape of \( f'(x) \).
- Geometrical interpretations of the derivative:
  - It is the slope of the line tangent to the graph of the function at the point.
  - It can be thought of as the slope of the function itself at the point, or more precisely, the slope of the line that best approximates the function at the point.

Zooming in on a curve
We have already seen how the secant line better and better approximates the tangent line as the interval used to estimate the derivative is made smaller and smaller.

By zooming in on a point on a curve, we find a second way of interpreting the derivative geometrically.
Example 7: Zoom in on the graph of \( y(t) = 4 + 3t - t^2 \) near \( t = 3 \). What happens to the curvature of the graph? Estimate the slope of the graph at \( t = 3 \).

On a very small scale, a smooth curve looks like a straight line. We call this line the best local linear approximation of the curve. The derivative of the curve is the slope of this line.

What the derivative of a function tells us about the function

Example 8: Let’s consider another example about population. A city’s population, in thousands, for various years is given in the table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>1970</th>
<th>1980</th>
<th>1990</th>
<th>2000</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(t) )</td>
<td>55</td>
<td>65</td>
<td>50</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>

The population increased in the 1970s, decreased in the 1980s, increased in the 1990s, and stayed the same in the 2000s. What was the sign of the average rate of change in the population for each of these decades?

In general:

- The rate of change of an increasing function is positive.
- The rate of change of a decreasing function is negative.
- The rate of change of a constant (unchanging) function is zero.

Furthermore, the faster a function is increasing or decreasing, the greater the magnitude of its rate of change.

The following questions refer to graphs (a)–(d) below. The graph (c) is the derivative of the graph (a). Likewise (d) is the derivative of (b). (I used formulas for the derivative of a function to make these graphs. You will learn these formulas in Chapter 3.)
Example 9: Determine the intervals on which $y$ is increasing or decreasing. What can you say about $y'$ on these intervals? What does $y'$ do when $y$ has a turning point? Repeat for $z$ and $z'$. 

Example 10: The zeros of $y'$ correspond to turning points of $y$. Does the same correspondence hold for $z'$ and $z'$?
Determining the shape of the derivative of a function. The following properties are helpful when you are trying to sketch the derivative of a function.

- Determine the intervals on which the function is increasing. The derivative is positive there. On intervals where the function is decreasing, the derivative is negative.
- Look for turning points and other places where the function is horizontal. The derivative has a horizontal intercept at those places.
- Look for places where the function is steepest. The derivative has a peak (if the function is increasing) or a valley (if the function is decreasing) at that place.

Example 11: Sketch the derivatives of the functions \( f(x) \) and \( g(x) \) shown below.

Determining the shape of a function from the graph of its derivative. We can also go in the other direction. Suppose you were told that the two graphs above were derivative functions of some unknown functions. How would you sketch those unknown functions?

Warning: The derivative of a function doesn’t change when you shift the function vertically up or down. Therefore, you can never determine the absolute vertical position of a function if all you know is its derivative. You can, however, determine its shape.
The following properties are helpful in sketching a function using the graph of its derivative.

- If the derivative is positive on an interval, the function is increasing on that interval. If the derivative is negative on an interval, the function is decreasing on that interval. If the derivative has a horizontal intercept at a point, the function is horizontal at that point.
- If the derivative is small (close to 0) the function is relatively flat. If the derivative is large, the function is steep.

**Example 12:** Sketch the functions $u(x)$ and $v(x)$ given their derivatives $u'(x)$ and $v'(x)$, keeping in mind that the vertical offset is arbitrary.

![Graphs of u'(x) and v'(x)](image)

**Newton and Leibniz notation for derivatives**

Calculus had two main discoverers, Isaac Newton and Gottfried Wilhelm Leibniz. Each invented his own notation, and both notations are still widely used. So far, we have been using Lagrange’s notation, which uses prime to denote the derivative. It is similar to Newton’s notation, which used dots instead of primes and is still commonly used in physics.

Newton and Lagrange’s notation has the following advantages:

- Writing the derivative of $f(x)$ as $f'(x)$ emphasizes that the derivative is also a function of $x$.
- It is easy to specify the point at which the derivative is taken: The derivative at $x = 2$ is written $f'(2)$. 
In Leibniz’s notation the derivative of $y$ is written $dy/dx$. Hence

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

When we approximate the derivative using the average rate of change formula, we use the symbol $\Delta$ to indicate the intervals used in the approximation. When we turn the estimate into an exact number by taking the limit, we change the symbol $\Delta$ to $d$.

Leibniz’s notation has the following advantages:

- It reminds us where the derivative comes from.
- It reminds us of the units the derivative has. For example, if $y$ is measured in meters and $x$ in seconds, then $dy/dx$ is measured in meters per second.
- Some of the shortcut formulas we will learn later are easier to remember in Liebniz notation.

On the other hand, it is a bit cumbersome to specify the point at which the derivative is being taken using Leibniz notation. If this is needed, the following notation is used:

$$\frac{dy}{dx} \bigg|_{x=2}$$

Warning: Although Leibniz’s notation makes the derivative look like a fraction, it is not correct to think of the derivative as the ratio of a quantity $dy$ to a quantity $dx$. We must always remember that the derivative is the limit of the ratio $\Delta y/\Delta x$ but is not itself a ratio.

Example 13: The value of a car $V$ in dollars is a decreasing function of the number of miles $x$ that it has been driven. Interpret $dV/dx$ and give its units. What is the meaning of $V'(50000)$?

Local linear approximation

Example 14: Suppose that $f(4) = 20$ and that $f'(4) = 3$. Find an approximate value of $f(4.1)$.

Solution: The slope of the graph of $f(x)$ at $x = 4$ is 3. Therefore, when $x$ increases by .1, $f(x)$ increases by approximately $3 \times .1 = .3$. Therefore $f(4.1) \approx 20 + .3 = 20.3$. Why is this only approximate?
This method of estimating \( f(4.1) \) is called **local linear approximation** since the answer is what you get by pretending that \( f(x) \) is a straight line with slope \( f'(4) = 3 \). In general, if we know \( f(x) \) and \( f'(x) \), then we can estimate the value of \( f(x + \Delta x) \) as follows:

\[
f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x.
\]

**Example 15:** Let \( f(x) = \sqrt{x} \). We compute that \( f(100) = 10 \). By the method from last time, you can compute that \( f'(100) = .05 \). (We will learn a much faster way to determine this in Chapter 3.) Estimate \( f(98) \) using local linear approximation. Then check how good this approximation is by computing \( f(98) \) with a calculator. Can you explain why the true value is lower than the approximation?

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**The relative rate of change**

We now connect our discussion, which has focused on the absolute rate of change, to the other kind of rate—the relative rate of change. Recall that we defined the relative change in \( f \) between time \( t_1 \) and time \( t_2 \) to be

\[
\frac{f(t_2) - f(t_1)}{f(t_1)}
\]

If \( t_1 \) and \( t_2 \) are spaced one unit apart, \( t_2 = t_1 + 1 \), then we can think of this as relative change per unit time. The rate \( r \) that we discussed in the context of exponential functions is defined this way.

More generally, we can divide the relative change by the time interval \( t_2 - t_1 \) to convert it to relative change per unit time or,

\[
\frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{f(t_2) - f(t_1)}{f(t_1)} \cdot \frac{1}{t_2 - t_1}.
\]

In the limit that \( t_2 \) approaches \( t_1 \), this ratio becomes the **relative rate of change**,

\[
\frac{f'(t_1)}{f(t_1)}.
\]

**Example 16:** Compute the relative rate of change for the function \( f(x) = e^{kx} \). To make this concrete, let \( k = 7 \) so that \( f(x) = e^{7x} \), and compute the relative rate of change at \( x = 3 \).
There is nothing special about \( x = 3 \). The relative rate of change of \( f(x) \) would have been the same for any \( x \), and would be equal to \( k \). Therefore the continuous rate, \( k \), of an exponential function is just the relative rate of change! Observe that \( k \) is the relative change per unit time in the limit that the time interval becomes infinitesimally small, whereas \( r \) is the relative change for a unit time interval.

- \( r \) = fractional change in \( f \) over a time interval of length 1;
- \( k \) = limiting value of the fractional change in \( f \) per unit time as the time interval approaches 0.

Exponential functions are functions with constant relative rate of change, but we can talk about the relative rate of change for any function. In general the relative rate of change will not be constant.

**Example 17:** The table below gives the U.S. gas price and its derivative as a function of time in months since January 1, 2012.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(t) )</td>
<td>3.36</td>
<td>3.50</td>
<td>3.78</td>
<td>4.00</td>
<td>3.89</td>
<td>3.73</td>
</tr>
<tr>
<td>( p'(t) )</td>
<td>.065</td>
<td>.210</td>
<td>.250</td>
<td>.055</td>
<td>-.135</td>
<td>-.235</td>
</tr>
</tbody>
</table>

Compute the relative rate of change at \( t = 1 \). Interpret \( p(1) \), \( p'(1) \), and the relative rate of change at time 1.

If the relative rate of change you computed at \( t = 1 \) had remained constant for an entire month, what would the relative change in the gas price have been for that month?