The Homotopy Groups of $K(\mathbb{S})$

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UIUC Topology Seminar

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Overview

Rognes calculated the homotopy groups of $K(\mathbb{S})$ at regular primes. What happens at irregular primes?

- Joint work with Andrew Blumberg
- Preprint arXiv:1408.0133

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1. Introduction and main result
2. Topological cyclic homology

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2. Topological cyclic homology
3. *K*-theory and étale cohomology

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1. Introduction and main result
2. Topological cyclic homology
3. K-theory and étale cohomology
4. Main theorem (reprise)

Waldhausen's Algebraic K-Theory of Spaces

Algebraic K-theory of spaces ties algebraic K-theory to differential and PL topology:

- $A(X) \simeq K(\mathbb{S}[X])$
- Smooth Whitehead space: $\Omega^\infty A(X) \simeq Q_+(X) \times Wh^{\mathrm{Diff}}(X)$
- Smooth stable concordance space: $\Omega Wh^{\mathrm{Diff}}(X) \simeq \mathscr{C}^{\mathrm{Diff}}(X)$

$$\mathscr{C}^{\mathrm{Diff}}(X) = \mathrm{colim}(C(X) \to C(X \times I) \to C(X \times I^2) \to \cdots)$$

- PL Whitehead space and PL stable concordance space

$$\Omega^\infty(K(\mathbb{S}) \wedge X_+) \to \Omega^\infty(A(X)) \to Wh^{\mathrm{PL}}(X)$$
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Linearization Map

map of ring spectra

Linearization map: $\mathbb{S} \longrightarrow \mathbb{Z}$

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$K(-)$ is functorial in maps of ring spectra

Linearization map: $\mathbb{S} \longrightarrow \mathbb{Z}$

$$K(\mathbb{S}) \longrightarrow K(\mathbb{Z})$$

Theorem (Waldhausen)

The linearization map $K(\mathbb{S}) \to K(\mathbb{Z})$ is a rational equivalence.

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Borel computed

$$\pi_*(K(\mathbb{Z});\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0,\ * \equiv 1 \bmod 4,\ * \neq 1 \\ 0 & \text{otherwise} \end{cases}$$

9	$\mathbb{Q}$
	0
	0
	0
5	$\mathbb{Q}$
	0
	0
	0
1	0
	$\mathbb{Q}$

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Linearization / Cyclotomic Trace Square

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Linearization map: $\mathbb{S} \longrightarrow \mathbb{Z}$

$$K(\mathbb{S}) \longrightarrow K(\mathbb{Z})$$

$$TC(\mathbb{S}) \longrightarrow TC(\mathbb{Z})$$

defined using equivariant stable homotopy theory

Topological cyclic homology TC

Theorem (Dundas)

The linearization/cyclotomic trace square becomes homotopy cartesian after p-completion.

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$$\text{cyclotomic trace} \quad \begin{array}{c} K \\ \downarrow \\ TC \end{array} \qquad \begin{array}{ccc} K(\mathbb{S}) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ TC(\mathbb{S}) & \longrightarrow & TC(\mathbb{Z}) \end{array}$$

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$$\begin{array}{ccc} K(\mathbb{S}) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ TC(\mathbb{S}) & \longrightarrow & TC(\mathbb{Z}) \end{array}$$

Consequence: Long exact sequence

$$\cdots \to \pi_n K(\mathbb{S})^\wedge_p \to \pi_n K(\mathbb{Z})^\wedge_p \oplus \pi_n(TC(\mathbb{S})^\wedge_p) \to \pi_n(TC(\mathbb{Z})^\wedge_p) \to \pi_{n-1} K(\mathbb{S})^\wedge_p \to \cdots$$

Theorem (Main Theorem)

The sequence $\pi_n K(\mathbb{S})^\wedge_p \to \pi_n K(\mathbb{Z})^\wedge_p \oplus \pi_n(TC(\mathbb{S})^\wedge_p) \to \pi_n(TC(\mathbb{Z})^\wedge_p)$ *is split short exact.* $(p > 2)$

Corollary: p-torsion is split short exact.

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n	$\pi_n K(\mathbb{S})$					
0	$\mathbb{Z}$					
1		$\mathbb{Z}/2$				
2		$\mathbb{Z}/2$				
3		$\mathbb{Z}/8 \times \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/2$			
4	0					
5	$\mathbb{Z}$					
6		$\mathbb{Z}/2$				
7		$\mathbb{Z}/16 \times \mathbb{Z}/3 \times \mathbb{Z}/5$	$\oplus\ \mathbb{Z}/2$			
8		$(\mathbb{Z}/2)^2$				$\oplus\ K_8(\mathbb{Z})$
9	$\mathbb{Z}\ \oplus$	$(\mathbb{Z}/2)^3$	$\oplus\ \mathbb{Z}/2$			
10		$\mathbb{Z}/2 \times \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/8 \times (\mathbb{Z}/2)^2$			
11		$\mathbb{Z}/8 \times \mathbb{Z}/9 \times \mathbb{Z}/7$	$\oplus\ \mathbb{Z}/2$	$\oplus\ \mathbb{Z}/3$		
12		$\mathbb{Z}/9$	$\oplus\ \mathbb{Z}/4$			$\oplus\ K_{12}(\mathbb{Z})$
13	$\mathbb{Z}\ \oplus$	$\mathbb{Z}/3$				
14		$(\mathbb{Z}/2)^2$	$\oplus\ \mathbb{Z}/4$	$\oplus\ \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/9$	
15		$\mathbb{Z}/32 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/5$	$\oplus\ (\mathbb{Z}/2)^2$			
16		$(\mathbb{Z}/2)^2$	$\oplus\ \mathbb{Z}/8 \times \mathbb{Z}/2$		$\oplus\ \mathbb{Z}/3$	$\oplus\ K_{16}(\mathbb{Z})$
17	$\mathbb{Z}\ \oplus$	$(\mathbb{Z}/2)^4$	$\oplus\ (\mathbb{Z}/2)^2$			
18		$\mathbb{Z}/8 \times \mathbb{Z}/2$	$\oplus\ \mathbb{Z}/32 \times (\mathbb{Z}/2)^3$		$\oplus\ \mathbb{Z}/3 \times \mathbb{Z}/5$	
19		$\mathbb{Z}/8 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/11$	$\oplus$ [64]			
20		$\mathbb{Z}/8 \times \mathbb{Z}/3$	$\oplus$ [128]		$\oplus\ \mathbb{Z}/3$	$\oplus\ K_{20}(\mathbb{Z})$
21	$\mathbb{Z}\ \oplus$	$(\mathbb{Z}/2)^2$	$\oplus$ [16]	$\oplus\ \mathbb{Z}/3$		
22		$(\mathbb{Z}/2)^2$	$\oplus$ [$2^?$]		$\oplus\ \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/691$

Topological Cyclic Homology

$TC(R)$ is built from the fixed points of $THH(R)$ and extra "cyclotomic" operators.

Theorem (Bökstedt-Hsiang-Madsen)

$TC(\mathbb{S})^{\wedge}_{p} \simeq (\mathbb{S} \vee \Sigma\mathbb{C}P^{\infty}_{-1})^{\wedge}_{p}$

$$\Sigma\mathbb{C}P^{\infty}_{-1} \to \Sigma\Sigma^{\infty}_{+}\mathbb{C}P^{\infty} \xrightarrow{\mathrm{Tr}_{\mathbb{T}}} \mathbb{S}$$

$$\Sigma^{\infty}_{+}\mathbb{C}P^{\infty} \simeq \mathbb{S} \vee \Sigma^{\infty}\mathbb{C}P \Longrightarrow (\mathbb{C}P^{\infty}_{-1})^{\wedge}_{p} \simeq \mathbb{S}^{\wedge}_{p} \vee \overline{\mathbb{C}P}^{\infty}_{-1}$$

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$TC(\mathbb{Z})$

Theorem (Bökstedt-Madsen) $(p > 2)$

$TC(\mathbb{Z})^{\wedge}_p[0,\infty) \simeq j \vee \Sigma j \vee \Sigma bu^{\wedge}_p$

$ku = KU[0,\infty)$

$bu = KU[2,\infty)$, $bu \simeq \Sigma^2 ku$

$j \to ku^{\wedge}_p \xrightarrow{\psi^k - 1} bu^{\wedge}_p$

$j \simeq L_{K(1)}\mathbb{S}[0,\infty)$

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$k \in \mathbb{Z}$ gen $\mathbb{Z}_p^{\times}$ top

gen $(\mathbb{Z}/p^2)^{\times}$

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$$ku^{\wedge}_{p} \simeq \ell \vee \Sigma^2 \ell \vee \cdots \vee \Sigma^{2p-4}\ell$$
$$\Sigma bu^{\wedge}_{p} \simeq \Sigma^3 \ell \vee \cdots \vee \Sigma^{2(p-2)-1}\ell \vee \Sigma^{2(p-1)-1}\ell \vee \Sigma^{2p-1}\ell$$

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$KU^\wedge_p \simeq L \vee \Sigma^2 L \vee \cdots \vee \Sigma^{2p-4} L$

$\pi_* ku = \mathbb{Z}[u] \quad |u| = 2$

$\pi_* \ell = \mathbb{Z}^\wedge_p[v_1] \quad |v_1| = 2p-2$

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$TC(\mathbb{Z})$

Theorem (Bökstedt-Madsen) $(p > 2)$

$$TC(\mathbb{Z})^{\wedge}_{p}[0,\infty) \simeq j \vee \Sigma j \vee \Sigma bu^{\wedge}_{p}$$

$$\Sigma^{2(p-1)} L \xrightarrow{\simeq} L$$

$ku = KU[0,\infty)$

$bu = KU[2,\infty)$, $bu \simeq \Sigma^2 ku$

$ku^{\wedge}_{p} \simeq \ell \vee \Sigma^2 \ell \vee \cdots \vee \Sigma^{2p-4}\ell$

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TC of the Linearization Map

Originally studied by Klein and Rognes

$\mathbb{S} \to \mathbb{Z}$

What we need is easy using $K(1)$-localization:

$$\begin{array}{ccc} TC(\mathbb{S})^{\wedge}_p & \longrightarrow & TC(\mathbb{Z})^{\wedge}_p \\ \simeq & & \simeq \\ \mathbb{S}^{\wedge}_p \vee \Sigma \mathbb{S}^{\wedge}_p \vee \overline{\mathbb{C}P}^{\infty}_{-1} & & j \vee \Sigma j \vee \bigvee(\Sigma^{2i-1}\ell) \vee \Sigma^{2p-1}\ell \end{array}$$

$\mathbb{S}_{(p)} \to \mathbb{Z}_{(p)}$ is $(2p-3)$-connected
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v_1 periodicity

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$L_{K(1)} ku = KU^{\wedge}_{p}$

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$$\downarrow$$

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v_1 periodicity
$\implies$ split surjection on π_* except $* \equiv 1 \pmod{2(p-1)}$

Some Facts About $K(\mathbb{Z})$, $K(\mathbb{Z}_p^\wedge)$

Theorem (Hesselholt-Madsen)

$$\begin{array}{ccc} K(\mathbb{Z})_p^\wedge & \longrightarrow & K(\mathbb{Z}_p^\wedge)_p^\wedge \\ \downarrow & & \downarrow \simeq \\ TC(\mathbb{Z})_p^\wedge[0,\infty) & \xrightarrow{\simeq} & TC(\mathbb{Z}_p^\wedge)_p^\wedge[0,\infty) \end{array}$$

$TC(\mathbb{Z})_p^\wedge[0,\infty) \simeq j \vee \Sigma j \vee \Sigma bu_p^\wedge, \quad L_{K(1)}TC(\mathbb{Z}) \simeq J \vee \Sigma J \vee \Sigma^{-1}KU$

$K(\mathbb{Z}_p^\wedge)_p^\wedge \to L_{K(1)}K(\mathbb{Z}_p^\wedge)$ induces isomorphism on π_* for $* > 1$.

Theorem (Quillen-Lichtenbaum Conjecture)

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Some Facts About $L_{K(1)}K(\mathbb{Z})$, $L_{K(1)}K(\mathbb{Z}_p^\wedge)$

Theorem (Thomason)

Let $R = \mathbb{Z}$ or $\mathbb{Z}_p^\wedge$. Let M be a p-torsion group or a pro-p-group.

$$\pi_{2q}(L_K(K(R)); M) \cong H^0_{\acute{e}t}(R[1/p]; M(q)) \oplus H^2_{\acute{e}t}(R[1/p]; M(q+1))$$
$$\pi_{2q-1}(L_K(K(R)); M) \cong H^1_{\acute{e}t}(R[1/p]; M(q))$$

Theorem (Poitou-Tate Duality)

Exact sequence

$$H^1_{\acute{e}t}(\mathbb{Z}[1/p]; \mathbb{Z}_p^\wedge(q)) \to H^1_{\acute{e}t}(\mathbb{Q}_p^\wedge; \mathbb{Z}_p^\wedge(q)) \to (H^1_{\acute{e}t}(\mathbb{Z}[1/p], \mathbb{Z}/p^\infty(1-q)))^*.$$

Look at $q = m(p-1)+1$

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A Fact About $H^1_{ét}$

Theorem

$H^1_{ét}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(-m(p-1))) = 0$

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Corollary

$K(\mathbb{Z})^\wedge_p \to TC(\mathbb{Z})^\wedge_p$ is surjective on π_n for $n \equiv 1 \pmod{2(p-1)}$.

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Corollary

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Actually, $K(\mathbb{Z})^\wedge_p$ splits $K(\mathbb{Z})^\wedge_p \simeq j \vee \Sigma^{2p-1}\ell \vee \text{rest}$

$$K(\mathbb{Z})^\wedge_p \qquad j \vee \text{rest} \vee \Sigma^{2p-1}\ell$$

$$\downarrow$$

$$TC(\mathbb{Z})^\wedge_p \qquad j \vee \Sigma j \vee (\bigvee \Sigma^{2i-1}\ell) \vee \Sigma^{2p-1}\ell$$

The Linearization/Cyclotomic Trace Square

$$\begin{array}{ccc} K(\mathbb{S})^\wedge_p & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathbb{S} \vee \Sigma\mathbb{S} \vee \overline{\mathbb{C}P}^\infty_{-1} & \longrightarrow & j \vee \Sigma j \vee (\bigvee \Sigma^{2i-1}\ell) \vee \Sigma^{2p-1}\ell \end{array}$$

Let $c = \mathrm{hofib}(\mathbb{S}^\wedge_p \to j)$ “coker j”

Theorem (Main Theorem)

$p\text{-}tors(K(\mathbb{S})) \cong p\text{-}tors(\mathbb{S}) \oplus p\text{-}tors(\Sigma c) \oplus p\text{-}tors(\overline{\mathbb{C}P}^\infty_{-1}) \oplus p\text{-}tors(K^{\mathrm{red}}(\mathbb{Z}))$

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n	$\pi_n K(\mathbb{S})$					
0	$\mathbb{Z}$					
1		$\mathbb{Z}/2$				
2		$\mathbb{Z}/2$				
3		$\mathbb{Z}/8 \times \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/2$			
4	0					
5	$\mathbb{Z}$					
6		$\mathbb{Z}/2$				
7		$\mathbb{Z}/16 \times \mathbb{Z}/3 \times \mathbb{Z}/5$	$\oplus\ \mathbb{Z}/2$			
8		$(\mathbb{Z}/2)^2$				$\oplus\ K_8(\mathbb{Z})$
9	$\mathbb{Z}\ \oplus$	$(\mathbb{Z}/2)^3$	$\oplus\ \mathbb{Z}/2$			
10		$\mathbb{Z}/2 \times \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/8 \times (\mathbb{Z}/2)^2$			
11		$\mathbb{Z}/8 \times \mathbb{Z}/9 \times \mathbb{Z}/7$	$\oplus\ \mathbb{Z}/2$	$\oplus\ \mathbb{Z}/3$		
12		$\mathbb{Z}/9$	$\oplus\ \mathbb{Z}/4$			$\oplus\ K_{12}(\mathbb{Z})$
13	$\mathbb{Z}\ \oplus$	$\mathbb{Z}/3$				
14		$(\mathbb{Z}/2)^2$	$\oplus\ \mathbb{Z}/4$	$\oplus\ \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/9$	
15		$\mathbb{Z}/32 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/5$	$\oplus\ (\mathbb{Z}/2)^2$			
16		$(\mathbb{Z}/2)^2$	$\oplus\ \mathbb{Z}/8 \times \mathbb{Z}/2$		$\oplus\ \mathbb{Z}/3$	$\oplus\ K_{16}(\mathbb{Z})$
17	$\mathbb{Z}\ \oplus$	$(\mathbb{Z}/2)^4$	$\oplus\ (\mathbb{Z}/2)^2$			
18		$\mathbb{Z}/8 \times \mathbb{Z}/2$	$\oplus\ \mathbb{Z}/32 \times (\mathbb{Z}/2)^3$		$\oplus\ \mathbb{Z}/3 \times \mathbb{Z}/5$	
19		$\mathbb{Z}/8 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/11$	$\oplus$ [64]			
20		$\mathbb{Z}/8 \times \mathbb{Z}/3$	$\oplus$ [128]		$\oplus\ \mathbb{Z}/3$	$\oplus\ K_{20}(\mathbb{Z})$
21	$\mathbb{Z}\ \oplus$	$(\mathbb{Z}/2)^2$	$\oplus$ [16]	$\oplus\ \mathbb{Z}/3$		
22		$(\mathbb{Z}/2)^2$	$\oplus$ $[2^?]$		$\oplus\ \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/691$

Multiplication on $\pi_* K(\mathbb{S})$ is Nilpotent

Suffices to consider rational part, *p*-torsion separately

Rational part is in odd degrees

Saw $K(\mathbb{Z})^\wedge_p \to TC(\mathbb{Z})^\wedge_p$ is surjective on π_n for $n \equiv 1 \pmod{2p-2}$.

Implies $\pi_n K(\mathbb{Z}) = 0$ for $n \equiv 0 \pmod{2p-2}$ (Poitou-Tate).

Implies $\pi_n \operatorname{hofib}(trc) = 0$ for $n \equiv 0 \pmod{2p-2}$.
Also for $p = 2$, $\pi_n \operatorname{hofib}(trc) = 0$ for $n \equiv 0 \pmod{8}$.

So $K(\mathbb{S})^\wedge_p \to TC(\mathbb{S})^\wedge_p$ injective on π_n for $n \equiv 0 \pmod{8p-8}$.

Suffices to see $\pi_* TC(\mathbb{S})^\wedge_p$ is nilpotent.

$TC(\mathbb{S})^\wedge_p \simeq \mathbb{S}^\wedge_p \vee \Sigma\mathbb{C}P^\infty_{-1}$