Introduction to Simplicial Complexes and Homology

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Applied Topology and High-Dimensional Data Analysis
Victoria, BC

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Outline

Basic definitions for simplicial complexes and the homology of simplicial complexes.

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1. Simplicial Complexes

2. Homology

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   - What are they?
   - What do they model?
   - Simplicial approximation
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   - What is a chain complex?
   - How do you get one?
   - Invariance theorem

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   - What is a chain complex?
   - How do you get one?
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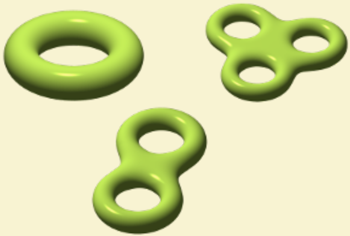
Example: Compact Surfaces



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Introduction to Simplicial Complexes

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Basic idea of a simplicial complex

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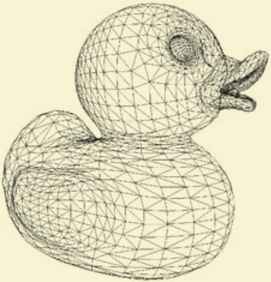

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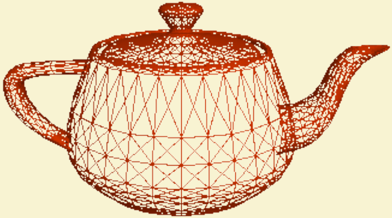
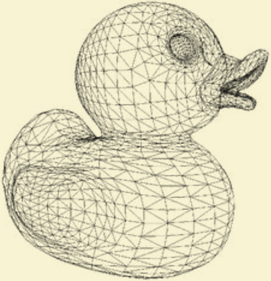
Introduction to Simplicial Complexes

Basic idea of a simplicial complex: Triangle Mesh / Triangulation





2-simplex = filled triangle

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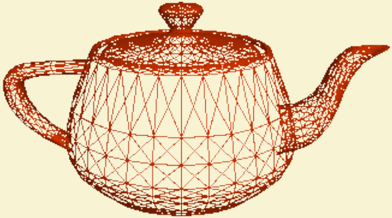

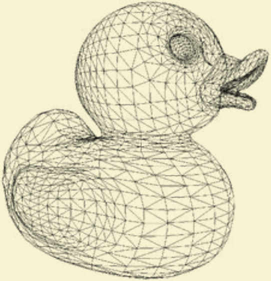

2-simplex = filled triangle with boundary three 1-simplexes

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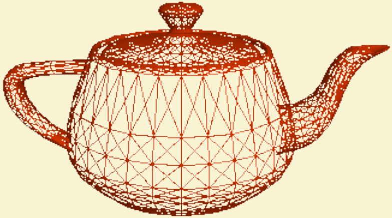

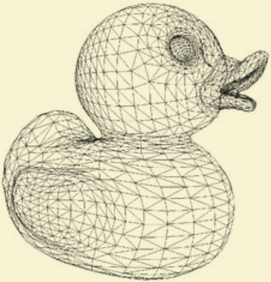

1-simplex = line segment between two 0-simplexes
2-simplex = filled triangle with boundary three 1-simplexes

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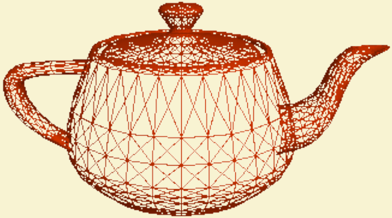

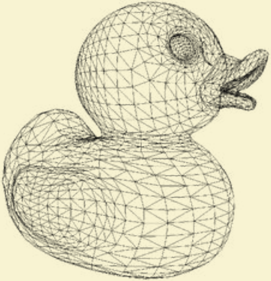

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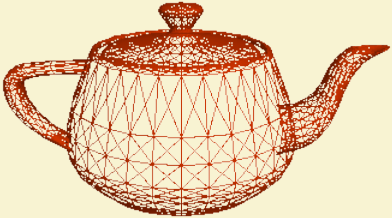

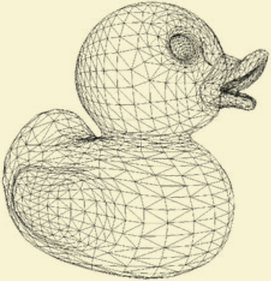

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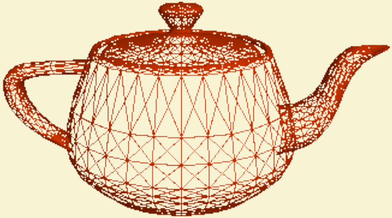

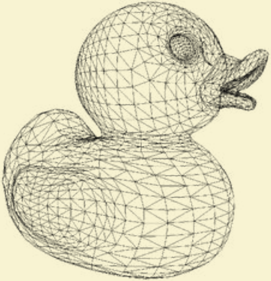

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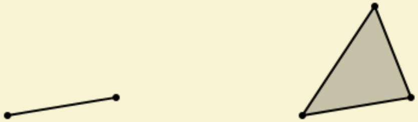
Geometric Simplexes

For $n+1$ points $V = \{v_0, \ldots, v_n\}$ in general position in a vector space



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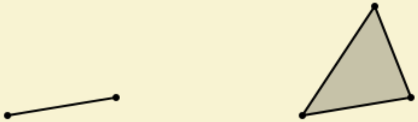
Barycentric coordinates

$$x = t_0 v_0 + t_1 v_1 + \cdots + t_n v_n \qquad 0 \leq t_i \leq 1, \quad \sum t_i = 1$$



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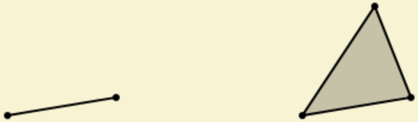

Standard *n*-simplex: Use $e_0, e_1, \ldots, e_n$ in $\mathbb{R}^{n+1}$

Geometric Simplicial Complex

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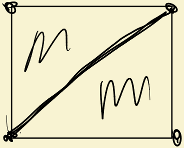

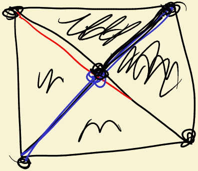

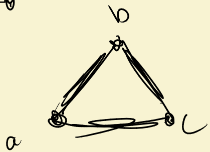

{a, b}
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Abstract Simplicial Complex

A geometric simplicial complex (X, V) is completely determined by the vertex set V and the set of simplexes σ_{A_α} in X.

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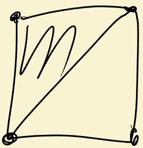
such that if $A \in S$ then every non-empty subset of A is in S.



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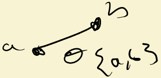
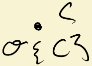
A map of simplicial complexes $(V, S) \to (V', S')$ is a function $f\colon V \to V'$ such that when $A \in S$, $f(A) \in S'$.

Geometric Realization

For a set V, let $\mathbb{R}\langle V\rangle$ be the vector space with basis V.

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If V is infinite, topologize $\mathbb{R}\langle V\rangle$ with the union topology for the finite subsets of V.

Geometric Realization

Definition

Let $K = (V, S)$ be an abstract simplicial complex. The geometric realization $|K|$ is the union of the simplexes in $\mathbb{R}\langle V\rangle$ spanned by the elements of S.

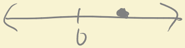
Visualize: $V = \{0, 1, \ldots, n\}$

$S =$ all finite non-empty subsets

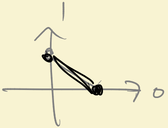
$K = (V, S)$

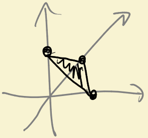
$|K| =$ standard n-simplex

$= 0$



$n = 1$





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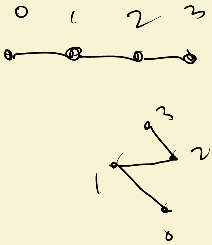

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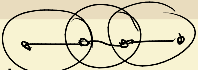

If $f\colon V \to \mathbb{R}^N$ is an injection onto a discrete subspace such that for each $A \in S$, $f(A)$ is in general position

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If $f\colon V \to \mathbb{R}^N$ is an injection onto a discrete subspace such that for each $A \in S$, $f(A)$ is in general position, and for each $A, B \in S$, $\sigma_{f(A)} \cap \sigma_{f(B)} = \sigma_{f(A \cap B)}$

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then $(X, f(V))$ is a geometric simplicial complex whose abstract simplicial complex is isomorphic to K via f.

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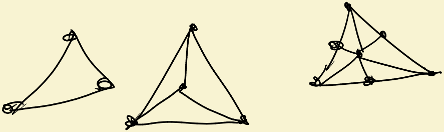
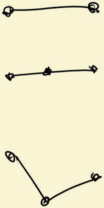
Moreover, the unique linear extension $\tilde{f}\colon \mathbb{R}\langle V\rangle \to \mathbb{R}^N$ induces a homeomorphism $|K| \to X$.

Subdivision





The Simplicial Approximation Theorem

Theorem

Let K and L be simplicial complexes and $f\colon |K| \to |L|$ a continuous map. There exists a subdivision K' of K and a simplicial map $g\colon K' \to L$ with $|g|$ homotopic to f

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Also a relative version for when f is already the geometric realization of a simplicial map on a subcomplex of K.

Approximation by Simplicial Complexes

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Spaces that are homeomorphic to simplicial complexes (examples)

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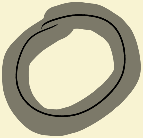
Spaces that are homotopy equivalent to simplicial complexes

- Euclidean neighborhood retracts



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Invariants of Simplical Complexes

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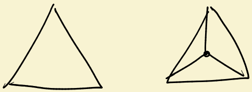
Counting simplexes

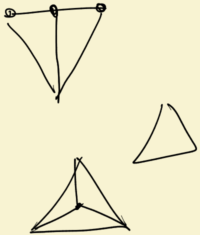


Invariants of Simplical Complexes

~~Counting simplexes~~

Alternating sum of number of simplexes

$$\#S_0 - \#S_1 + \#S_2 - \cdots$$



Euler characteristic

Invariants of Simplical Complexes

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Euler characteristic

Powerful enough to classify compact surfaces, almost

Euler Characteristic and Compact Surfaces

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Image credit:

toruses: Oleg Alexandrov / Wikipedia

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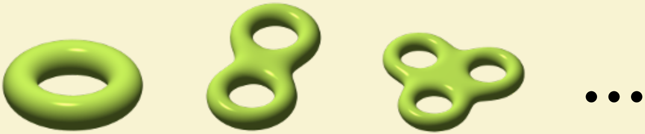

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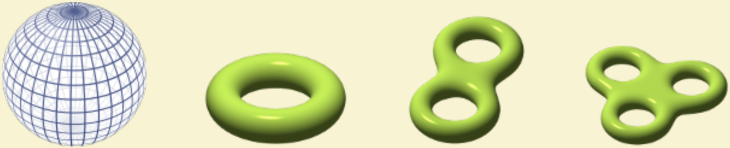
toruses: Oleg Alexandrov / Wikipedia

Euler Characteristic and Compact Surfaces

Compact Surfaces



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Euler Characteristic and Compact Surfaces

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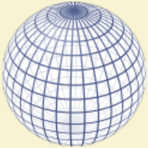


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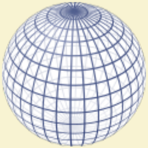

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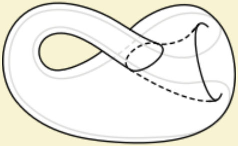

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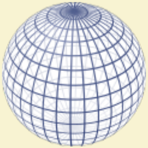



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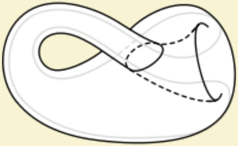
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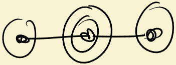
Components

define an equiv rel on vertices
0-simplices

$a \sim b$ if there exists a 1-simplex $\{a, b\}$.

(generated by this)



Components

Components

H_1

Free abelian gp on 1-simplexes

subgp Subset where 0-th vertex = 1-st vertex

free abelian gp on 0-simplices

quotient by for any 2-simplex $[v_0, v_1, v_2]$

$[v_0, v_2] \sim [v_0, v_1] + [v_1, v_2]$

$[v_0, v_1] - [v_0, v_2] + [v_1, v_2] = 0$

The Linear Algebra of Faces

Let K be a simplicial complex.

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An algebraic object like this is called a chain complex

Chain Complexes and Homology

Definition

A chain complex is a sequence of abelian groups (or vector spaces) $C_0, C_1, \ldots,$ and homomorphisms $d_1\colon C_1 \to C_0$, $d_2\colon C_2 \to C_1, \ldots,$ such that $d_n \circ d_{n+1} = 0$.

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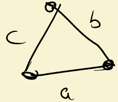
Definition

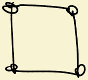
$H_n = Z_n/B_n$ is called the *n*-th homology group.

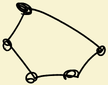
Examples

H_0 = free abelian gp on components

H_1







(no 2-simplexes)

$H_1 = \mathbb{Z}$ on $\underline{a - b + c}$

Fill in with a 2-simplex

$H_1 = 0$

H_2 $\quad \partial$ 3-simplex $\cong S^2$

Homomorphisms and Chain Homotopies

A homomorphism of chain complexes $f_* \colon C_* \to C'_*$ consists of homomorphisms $f_n \colon C_n \to C'_n$ such that $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$

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Given homomorphisms f_* and g_*, a chain homotopy from f_* to g_* consists of homomorphisms $s_n \colon C_n \to C'_{n+1}$ such that

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Chain homotopic maps induce the same map on homology

Subdivision

The Invariance Theorem

Let K and L be simplicial complexes and let $f, g\colon K \to L$ be maps of simplicial complexes. If $|f|$ and $|g|$ are homotopic, then f and g are chain homotopic and induce the same homomorphism on homology.

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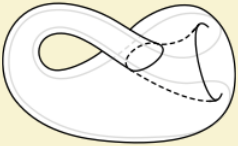
Homology of Compact Surfaces



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Klein bottle: https://commons.wikimedia.org/wiki/File:Surface_of_Klein_bottle_with_traced_line.svg

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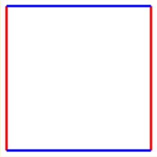

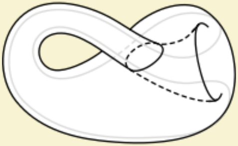

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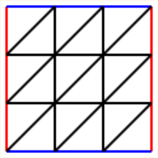

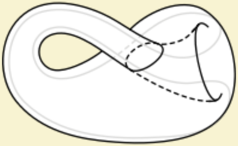

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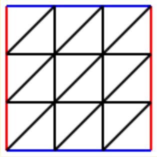

$$H_0 \cong \mathbb{Z}$$
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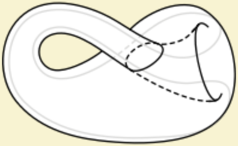

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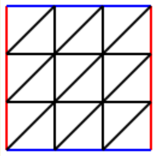

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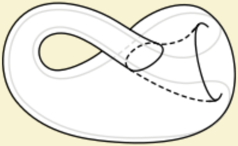

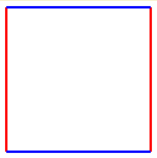

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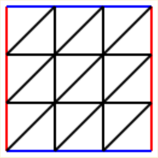

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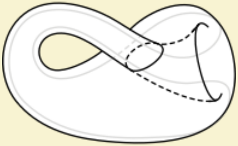

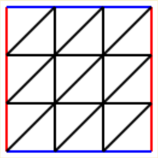

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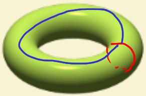

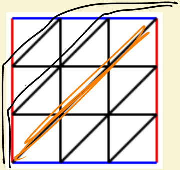

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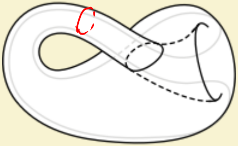

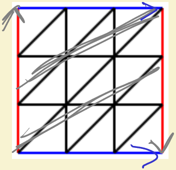

$$H_0 \cong \mathbb{Z}$$
$$H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2$$
$$H_2 \cong 0$$

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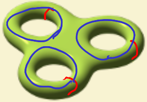
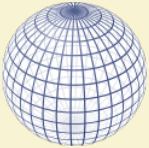
Homology of Compact Surfaces



$H_0 \cong \mathbb{Z}$
$H_1 \cong \mathbb{Z}^{2n}$
$H_2 \cong \mathbb{Z}$

(genus n)



$H_0 \cong \mathbb{Z}$
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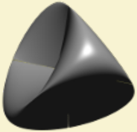

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Lots More to Say

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1. Simplicial Complexes

2. Homology

Lots More to Say

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   - Star neighborhoods
   - Simplicial approximations
   - Contiguity

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   - Handle attachment
   - Intersections / Poincaré duality / cohomology
3. Intermediate topics

Lots More to Say

1. Simplicial Complexes
   - Star neighborhoods
   - Simplicial approximations
   - Contiguity
2. Homology
   - Pairs, long exact sequences
   - Mayer-Vietoris
   - Handle attachment
   - Intersections / Poincaré duality / cohomology
3. Intermediate topics
   - Homotopy theory: homotopy groups, fibrations, cofibrations
   - Spectral sequences